# A duality-based path-following semismooth Newton method for elasto-plastic contact problems

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**Abstract**

A Fenchel dualization scheme for the one-step time-discretized contact problem of quasi-static elasto-plasticity with combined kinematic-isotropic hardening is considered. The associated path is induced by a coupled Moreau-Yosida / Tichonov regularization of the dual problem. The sequence of solutions to the regularized problems is shown to converge strongly to the optimal displacement-stress-strain triple of the original elasto-plastic contact problem in the space-continuous setting. This property relies on the density of the intersection of certain convex sets which is shown as well. It is also argued that the mappings associated with the resulting problems are Newton- or slantly differentiable. Consequently, each regularized subsystem can be solved mesh-independently at a local superlinear rate of convergence. For efficiency purposes, an inexact path-following approach is proposed and a numerical validation of the theoretical results is given.

*Keywords:* elasto-plastic contact, variational inequality of the 2nd kind, Fenchel duality, Moreau-Yosida/Tichonov regularization, path-following, semismooth Newton

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## 1. Introduction

In this paper we consider the quasi-static elasto-plasticity model with an associative flow law (sometimes called Prandtl-Reuss normality law) and von Mises hardening under the small strain assumption set forth in [22]. First investigations of the elasto-plastic problem from a mathematical point of view can be found in [16, 33], where [33] includes existence for the fully continuous case. Numerical analysis of the semi-discrete and fully-discrete versions can be found, for example, in [2, 22]. Appropriate discretization schemes for plasticity problems with hardening have been investigated extensively in the recent past. Here we only mention [3, 10, 9, 43] for adaptive finite element methods. Concerning numerical solution methods, we refer to the multigrid approach in [47], various generalized Newton methods in finite dimensions [12, 20, 42, 47, 48], including the standard return mapping algorithm in [44] as well as interior point strategies, cf. e.g. [37].

A general introduction to elastic contact problems including corresponding numerical approaches can be found in the monographs [31, 41], and multigrid methods for elastic contact are analyzed, e.g.,

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in [35] and [36, 38], where the latter references are devoted to two-body contact. For the treatment of elastic friction problems we refer to [13, 38] as well as to the efficient active set algorithm proposed in [32]. Subspace correction methods for variational inequalities of the second kind with application to frictional contact have been investigated in [5]. In [12, 21] plastic material behavior is incorporated in addition to the contact constraints. In the latter references the elasto-plastic friction problem is reformulated utilizing a nonlinear complementarity problem (NCP) function yielding a nonsmooth system which can be solved efficiently by applying a generalized Newton method in a discrete framework provided a set of damping parameters is chosen appropriately.

While some attention has been paid to infinite-dimensional methods in linear elasticity with (frictional) contact [39, 45], elasto-plastic problems are still less researched. Among the few available references we mention [8] for domain decomposition methods leading to a linear rate of convergence. The approach to plasticity problems without contact constraints in [20], however, turns out to be problematic as far as function space convergence of the employed semismooth Newton (SSN) solver is concerned. In fact, due to the lack of a sufficient norm gap between domain and image space of the mapping involved in the underlying nonsmooth system, generalized differentiability in the sense of [30] does not hold true. The resulting lack of a well-defined infinite-dimensional generalized Newton iteration usually results in a mesh-dependent solver.

In the present paper, we introduce a path-following semismooth Newton method which admits a rigorous convergence analysis in the continuous setting. For this purpose, we study a regularized version of the Fenchel-dual problem of the underlying elasto-plastic contact problem with the regularization parameter inducing a dual path to the solution of the original problem. Each path-problem can be solved at a local superlinear rate and in a mesh-independent way upon discretization.

## 2. Problem formulation

The starting point of our analysis is the small-strain elasto-plastic contact problem in the displacement $u$, the plastic strain $p$ and a set of internal variables $\xi$ which model the evolution of a body subject to given applied forces. The body is represented by a bounded domain $\Omega \subset \mathbb{R}^N, N = 2, 3$, with $N^{0,1}$-property [49] and it adheres to a fixed part $\Gamma_d \subset \partial\Omega$ with positive surface measure. We further denote by $\Gamma_n \subset \partial\Omega \backslash \Gamma_d$ some relatively open part of the boundary where a given surface load $g \in L^2(\Gamma_n)$ is applied. A given volume force density is denoted by $f \in L^2(\Omega)$. The elasto-plastic behavior at a material point $x \in \Omega$ is determined by a given yield criterion leading to a dissipation functional which typically is nonsmooth, lower semicontinuous (l.s.c.) and convex [22]. Often, the displacement of the body is restricted by a given rigid obstacle giving rise to an elasto-plastic contact problem. Therefore we fix a set $\Gamma_c \subset \partial\Omega$ which potentially contains the contact region with the obstacle. We emphasize here that the approach presented in this work does not hinge on $\Gamma_c \neq \emptyset$. To measure the gap between $\Omega$ and the obstacle we use a given function

$$\psi \in Z := H^{1/2}(\Gamma_c) \text{ with } \psi \geq 0 \text{ almost everywhere (a.e.) on } \Gamma_c;$$

see [41]. For the time being we neglect frictional forces such that in terms of the variational formulation, we incorporate the contact constraint by a kinematic non-penetration condition on the displacement $u$:

$$\tau_n u \leq \psi \text{ on } \Gamma_c, \tag{2.1}$$

where $\tau_n : [H^1_{0,\Gamma_d}(\Omega)]^N \to Z, u \mapsto (\tau|_{\Gamma_c}(u)) \cdot n$ denotes the normal trace mapping restricted to $\Gamma_c$. For analytical reasons we assume that $\Gamma_c$ is relatively open with $N^{1,1}$-property and $C^\infty$-boundary $\partial\Gamma_c$. For simplicity and without loss of generality we further stipulate

$$\overline{\Gamma}_c \subset \Sigma, \tag{2.2}$$

where $\Sigma$ denotes the interior of $\partial\Omega\backslash\Gamma_d$ in $\partial\Omega$, to avoid working with the space $H^{1/2}_{00}(\Gamma_c)$. Concerning the splitting of the boundary we further assume

$$\partial\Omega = \overline{\Gamma}_c \cup \overline{\Gamma}_n \cup \overline{\Gamma}_d, \quad \Gamma_c \cap \Gamma_n \cap \Gamma_d = \emptyset, \quad \partial\Sigma \in C^\infty.$$

To formulate the quasi-static problem, we first fix the notation which is loosely based on the monograph by Han and Reddy [23]. We endow the Hilbert spaces

$$V := [H^1_{0,\Gamma_d}(\Omega)]^N, \quad Q := [L^2(\Omega)]^{N\times N}_{\mathrm{sym}}$$

with the usual scalar products. In this context, $\mathbb{C}(x) \in \mathbb{R}^{N\times N\times N\times N}, \mathbb{C}_{ijkl} \in L^\infty(\Omega)$, denotes the fourth-order elasticity tensor which is assumed to be symmetric, i.e. $\mathbb{C}_{ijkl} = \mathbb{C}_{klij} = \mathbb{C}_{jikl}$ and pointwise stable, i.e. $\exists C > 0$ with

$$\mathbb{C}(x)\sigma : \sigma \geq C|\sigma|^2_F \quad \forall\, \sigma \in \mathbb{R}^{N\times N}_{\mathrm{sym}} \text{ and a.e. } x \in \Omega,$$

where $A : B = \sum_{i,j=1\dots N} a_{ij} \cdot b_{ij}$ for $A, B \in \mathbb{R}^{N\times N}$. Analogous properties are supposed to be fulfilled by the hardening modulus $\mathbb{H}(x) \in \mathbb{R}^{m\times m}$. The symmetric part of the displacement gradient is denoted by $\varepsilon(u)$, i.e.,

$$\varepsilon(u)(x) = \tfrac{1}{2}(\nabla u(x) + \nabla u(x)^\top).$$

Further, $\mathrm{tr}(\sigma) := \sum_{i=1}^N \sigma_{ii}$ stands for the matrix trace operator. The plastic incompressibility condition on $p$ gives rise to the closed subspace $Q_0$ of $Q$ defined by

$$Q_0 := \{q \in [L^2(\Omega)]^{N\times N}_{\mathrm{sym}} : \mathrm{tr}(q) = 0 \text{ a.e. in } \Omega\}$$

which inherits the scalar product of $Q$.

**Quasi-static elasto-plastic contact problem.** Given some material-dependent l.s.c., convex and proper yield functional $\phi : \mathbb{R}^{N\times N}_{\mathrm{sym}} \times \mathbb{R}^m \to \mathbb{R} \cup \{+\infty\}$, the underlying elasto-plastic contact problem with a linear hardening law consists of seeking $(u, p, \xi)(t) \in V \times Q_0 \times L^2(\Omega)^m, t \in [0, T]$, with $(u, p, \xi)(0) = 0$, such that

$$u = 0 \quad \text{on } \Gamma_d, \tag{2.3}$$
$$\sigma n = g \quad \text{on } \Gamma_n, \tag{2.4}$$
$$\operatorname{div} \sigma = -f, \tag{2.5}$$
$$\varepsilon(u) = \mathbb{C}^{-1}\sigma + p, \tag{2.6}$$
$$(\sigma, -\mathbb{H}\xi) \in K := \{(\tilde{\sigma}, \tilde{\chi}) : \phi(\tilde{\sigma}, \tilde{\chi}) \leq 0\}, \tag{2.7}$$
$$(\dot{p}, \dot{\xi}) \in N_K(\sigma, -\mathbb{H}\xi), \tag{2.8}$$
$$\tau_T\sigma = 0,\ \tau_{nn}\sigma \leq 0, \tau_{nn}\sigma(\tau_n u - \psi) = 0,\ \tau_n u \leq \psi \quad \text{on } \Gamma_c, \tag{2.9}$$

for a.e. $t \in [0,T]$, where $N_K(\tilde\sigma,\tilde\chi)$ denotes the normal cone to the convex set $K$ at $(\tilde\sigma,\tilde\chi)$. Furthermore, $\tau_{nn}\sigma := (\tau_n\sigma)^\top n$, and $\tau_T\sigma := \tau_n\sigma - (\tau_{nn}\sigma)n$ denotes the tangential trace on $\Gamma_c$, and $(\dot p,\dot\xi)$ represent the derivative in time. Note that (2.6)-(2.8) determine the plasticity behavior and (2.9) represents the complementarity conditions of contact; for details cf. [23, 41].

**Incremental formulation.** An implicit Euler discretization of the time derivatives appearing in the associative flow law (2.8) leads to the following weak form of the incremental problem:

$$\begin{cases} \min & \tilde J(u,p,\xi) \quad \text{over } (u,p,\xi) \in V \times Q \times L^2(\Omega)^m \\ \text{subject to (s.t.)} & \tau_n u \le \psi \text{ on } \Gamma_c, \end{cases} \tag{2.10}$$

with

$$\begin{aligned} \tilde J(u,p,\xi) := \tfrac12 \int_\Omega & \mathbb{C}(\varepsilon(u)-p) : (\varepsilon(u)-p) + \xi : \mathbb{H}\xi \,\mathrm{dx} \\ &+ \int_\Omega \chi_K^*(p-p_0,\xi-\xi_0)\,\mathrm{dx} \\ &\qquad - \int_\Omega f\cdot u\,\mathrm{dx} - \int_{\Gamma_n} g\cdot u\,\mathrm{dx}, \end{aligned}$$

where $\chi_K^*$ denotes the convex conjugate of the characteristic function $\chi_K$ of the convex set $K$ and $p_0,\xi_0$ denote the states of the variables from the preceding time instance.

**Combined linearly isotropic-kinematic hardening with the von Mises yield condition.** For combined isotropic-kinematic hardening it holds that $\xi = [p,\eta] \in \mathbb{R}_0^{n\times n} \times \mathbb{R}$, $\mathbb{H}(p,\eta) = k_1 p + k_2\eta$ with $k_1,k_2 \ge 0$, and the associated von Mises yield function is defined by

$$\phi(\sigma,[a,g]) := |\operatorname{dev}\sigma + a|_F + g - \sigma_y + \chi_{\mathbb{R}_0^-}(g),\ [a,g] \in \mathbb{R}_0^{n\times n}\times\mathbb{R}, \tag{2.11}$$

with some material-dependent yield stress $\sigma_y > 0$, cf. [23]. In this case, a variable shift replacing $(p-p_0)$ by $p$ in (2.10), leads to the problem

$$\begin{cases} \min & J(u,p) \quad \text{over } (u,p) \in V \times Q_0 \\ \text{s.t.} & \tau_n u \le \psi \text{ on } \Gamma_c \end{cases} \tag{2.12}$$

with

$$J(u,p) := \tfrac12 \int_\Omega \mathbb{C}(\varepsilon(u)-p) : (\varepsilon(u)-p) + \bar k\, p : p\,\mathrm{dx} + \int_\Omega \beta |p|_F \,\mathrm{dx} + l(u,p),$$

where $\beta = (\sigma_y + k_2\eta_0) \in L^2(\Omega)$ with $\beta \ge \sigma_y$ a.e. in $\Omega$, $\bar k = (k_1+k_2) \in L^2(\Omega)$, and a linear functional

$$l(u,p) := -\int_{\Gamma_n} gu\,\mathrm{ds} - \int_\Omega fu + k_1\, p_0 : p - \mathbb{C}(\varepsilon(u)-p) : p_0\,\mathrm{dx},$$

with $l \in (V\times Q_0)^*$, the topological dual space to $V \times Q_0$. Note that (2.12) is equivalent to an elliptic variational inequality of the mixed (i.e. first and second) kind. Writing

$$\begin{aligned} y &:= (u,p) \in Y := V \times Q_0, \\ p &=: \Pi_{Q_0}(u,p),\ \Pi_{Q_0} \in \mathcal{L}(Y,Q_0), \\ a([u,p],[\tilde u,\tilde p]) &:= \int_\Omega \mathbb{C}(\varepsilon(u)-p) : (\varepsilon(\tilde u)-\tilde p) + \bar k\, p : \tilde p\,\mathrm{dx}, \end{aligned}$$

yields a more compact form of $J : Y \to \mathbb{R}$:

$$J(y) = \tfrac{1}{2}\langle Ay, y\rangle_{(Y^*,Y)} + l(y) + \int_\Omega \beta \cdot |\Pi_{Q_0} y|_F \,\mathrm{dx}, \tag{EP}$$

where $A \in \mathcal{L}(Y, Y^*)$ is the linear and continuous operator from $Y$ to its topological dual $Y^*$ associated to the bilinear form $a : Y \times Y \to \mathbb{R}$. We note that $a$ is $Y$-elliptic if $\operatorname{essinf}_\Omega \bar{k} > 0$, cf. [22]. Standard arguments then show that (2.12) admits a unique solution $\bar{y} = (\bar{u}, \bar{p}) \in Y$. The condition on $\bar{k}$ is always supposed to be fulfilled, otherwise we would leave the framework of hardening plasticity for a problem of perfect plasticity which requires a different functional analytic setting, cf. [14]. However, the resulting problem may be approximated consistently by a sequence of plasticity problems with vanishing hardening [6].

**Remark.** Using Moreau's theorem, (EP) can be further reduced to a (Fréchet) differentiable problem in the displacement only, cf. [20]. However, the resulting optimality condition is not eligible to Newton differentiation (in the sense of [30]) in infinite dimensions which may result in mesh-dependent convergence of an associated generalized Newton scheme. While the Newton differentiability of the stationarity system is always given in finite dimensions, the spatially continuous case requires a certain norm gap which is indispensable for the Newton differentiation of the involved composed max-function, cf. [27, 28] or section 6 for related issues. Such an integrability gap can never be achieved without further regularization.

## 3. Fenchel duality for the elasto-plastic contact problem

For numerical purposes it turns out that the Fenchel dual problem to (2.12) is favorable in the sense that, upon regularization, it can be solved efficiently by semismooth Newton techniques.

In order to establish a compact set-up for the application of the Fenchel duality theory, the elasto-plastic contact problem (2.12) will be rewritten in the form

$$\min F(y) + G(\Lambda y), \qquad \text{over } y \in Y, \tag{EPC}$$

with a Gâteaux-differentiable function $F$, a l.s.c., proper, and convex function $G$ and a linear and continuous operator $\Lambda$. In fact, we define $F : Y \to \mathbb{R}$ by

$$F(y) := \tfrac{1}{2}\langle Ay, y\rangle_{(Y^*,Y)} + l(y).$$

Further, we denote the convex cone associated to the constraint (2.1) by

$$K_1 := \{z \in Z : z \leq 0 \text{ a.e. on } \Gamma_c\}$$

and define $G : Z \times L^2(\Omega)^d \to \mathbb{R} \cup \{+\infty\}$ by

$$G(z, q) := G_1(z) + G_2(q) := \chi_{\psi + K_1}(z) + \int_\Omega \beta |q|_2 \,\mathrm{dx}\,.$$

Moreover, we set

$$\Lambda := \begin{bmatrix} \tau_n & 0 \\ 0 & M^{1/2}P^{-1} \end{bmatrix} \in \mathcal{L}(Y, Z \times L^2(\Omega)^d),$$

where $\chi_{\psi+K_1}$ is the indicator function of the set $\psi + K_1$, and

$$P : (L^2(\Omega)^d, \|\,.\,\|_{L^2(\Omega)^d}) \to (Q_0, \|\,.\,\|_{Q_0}),$$

with $d = \frac{N(N+1)}{2} - 1$, denotes the canonical parametrization given by

$$[q_1, q_2] \overset{P}{\mapsto} \begin{bmatrix} q_1 & q_2 \\ q_2 & -q_1 \end{bmatrix}; \quad [q_1, q_2, q_3, q_4, q_5] \overset{P}{\mapsto} \begin{bmatrix} q_1 & q_3 & q_4 \\ q_3 & q_2 & q_5 \\ q_4 & q_5 & -(q_1+q_2) \end{bmatrix}; \tag{3.1}$$

for $N = 2, 3$, respectively. The symmetric positive definite matrix $M$ is defined by

$$M = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}; \quad M = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix};$$

for $n = 2, 3$, respectively, such that $Pp : Pq = Mp \cdot q \quad \forall\, p, q \in \mathbb{R}^d$.

This setting differs from the one presented in [45] mainly by the choice of the operator $\Lambda$ which entails a slightly different interpretation of the dual variable $q$, cf. (3.8). We next compute and analyze the dual problem to (EPC).

**Computation of the Fenchel conjugates.** The convex conjugate $F^* : Y^* \to \mathbb{R}$ of $F : Y \to \mathbb{R}$ is given by

$$F^*(y^*) = \tfrac{1}{2}\langle y^* - l, A^{-1}(y^* - l)\rangle_{(Y^*,Y)}.$$

For the nondifferentiable part $G$ we obtain

$$G^* : Z^* \times L^2(\Omega)^d \to \mathbb{R} \cup \{+\infty\}, \quad G^*(z^*, q) = G_1^*(z^*) + G_2^*(q),$$

with

$$G_2^* : L^2(\Omega)^d \to \mathbb{R} \cup \{+\infty\}, \quad G_2^*(q) = \chi_{K_2}(q),$$

where $K_2 := \{q \in L^2(\Omega)^d : |q|_2 \le \beta \text{ a.e. in } \Omega\}$, and

$$G_1^* : Z^* \to \mathbb{R} \cup \{+\infty\}, \quad G_1^*(z^*) = \sup_{z \in K_1 + \psi} \langle z^*, z\rangle = \chi_{K_1^*}(z^*) + \langle z^*, \psi\rangle,$$

where it is understood that

$$\begin{aligned} K_1^* := Z_+^* &= \{z^* \in Z^* : z^* \ge 0\} \\ &= \{z^* \in Z^* : \langle z^*, z\rangle \ge 0 \;\; \forall\, z \in Z \text{ with } z \ge 0 \text{ a.e. on } \Gamma_c\}. \end{aligned} \tag{3.2}$$

The dual problem to (EPC) is given by

$$\sup -F^*(-\Lambda^*[z^*, q]) - G^*(z^*, q) \quad \text{over } [z^*, q] \in Z^* \times L^2(\Omega)^d, \tag{D}$$

which can be equivalently expressed as

$$\begin{cases} -\inf & F^*(\Lambda^*[z^*, q]) - \langle z^*, \psi\rangle \quad \text{over } [z^*, q] \in Z^* \times L^2(\Omega)^d \\ \text{s.t.} & z^* \le 0, \\ & |q|_2 \le \beta \text{ a.e. in } \Omega. \end{cases}$$

Note the sign change in the dual variables and that the first inequality constraint has to be understood in the sense of (3.2).

Since $K_1 + \psi$ has empty interior, a generalized Slater condition fails to hold. Hence the Fenchel duality theorem in its usual version [17] is not applicable. However, in our special situation we are still able to preclude the presence of a duality gap.

**Proposition 3.1** (Duality). *There is no duality gap, i.e. it holds that*

$$\inf\,(\text{EPC}) = \sup\,(\text{D}).$$

*Moreover, there exists a unique solution $(\bar{z}, \bar{q}) \in Z^* \times L^2(\Omega)^d$ to the dual problem.*

Proof. We make use of [4, Theorem 1, Chapter 4.6], and need to show that

$$0 \in \operatorname{int}\left(\Lambda^* \operatorname{dom} G^* + \operatorname{dom} F^*\right). \tag{3.3}$$

As $F^*$ is finite everywhere, we have $\operatorname{dom} F^* = Y^*$. Further, $\operatorname{dom} G^* \neq \emptyset$ implies $\Lambda^* \operatorname{dom} G^* + \operatorname{dom} F^* = Y^*$ such that (3.3) is always satisfied. It follows that no duality gap occurs.
Regarding existence and uniqueness of a solution to (D) we notice that the objective function is continuous and strictly convex since $F^*$ is strongly convex and $\Lambda^*$ is injective by the surjectivity of $\tau_n$, cf. (2.2). Moreover, coercivity of the objective function follows from ellipticity of the bilinear form associated to $A^{-1}$. Indeed, with $\kappa > 0$ denoting the corresponding ellipticity constant, it follows that

$$\begin{aligned}
&F^*(\Lambda^*[z^*, q]) - \langle z^*, \psi\rangle \\
&= \tfrac{1}{2}\langle \Lambda^*[z^*, q] - l, A^{-1}(\Lambda^*[z^*, q] - l)\rangle_{(Y^*,Y)} - \langle z^*, \psi\rangle \\
&\geq \tfrac{\kappa}{2}\|\Lambda^*[z^*, q]\|^2_{Y^*} - \|\Lambda A^{-1} l + [\psi, 0]\|\|[z^*, q]\|_{Z^* \times L^2(\Omega)^d} + \tfrac{\kappa}{2}\|l\|^2 \\
&\geq \tfrac{\kappa}{2\|\Lambda^{-*}\|^2}\|[z^*, q]\|^2_{Z^* \times L^2(\Omega)^d} - \|\Lambda A^{-1} l + [\psi, 0]\|\|[z^*, q]\|_{Z^* \times L^2(\Omega)^d} + \tfrac{\kappa}{2}\|l\|^2,
\end{aligned}$$

where the last estimate follows from the fact that $\Lambda^*$ has a bounded inverse on its (closed) range owing to the closed range theorem. □

**Optimality conditions.** By the absence of a duality gap (Proposition 3.1), the solution $\bar{y} = [\bar{u}, \bar{p}]$ of the primal problem (EPC) can be recovered from the solution $[\bar{z}, \bar{q}]$ of (D) from

$$\begin{aligned}
&\Lambda^*[\bar{z}, \bar{q}] = A\bar{y} + l, \\
&-[\bar{z}, \bar{q}] \in \partial G(\Lambda \bar{y}).
\end{aligned} \tag{P-D}$$

Due to (3.3), we may characterize the solution $[\bar{z}, \bar{q}] \in Z^* \times L^2(\Omega)^d$ by the existence of $\bar{\lambda} = [\bar{\mu}, \bar{\nu}] \in Z \times L^2(\Omega)^d$ satisfying

$$\Lambda A^{-1}\Lambda^*[\bar{z}, \bar{q}] - \Lambda A^{-1} l - [\psi, 0] + \bar{\lambda} = 0, \tag{OC1}$$

$$\bar{z} \leq 0, |\bar{q}|_2 \leq \beta \text{ a.e. in } \Omega, \tag{OC2}$$

$$\langle \bar{\mu}, z^* - \bar{z}\rangle \leq 0,\ (\bar{\nu}, q - \bar{q}) \leq 0\ \ \forall\, z^* \leq 0,\ \forall\, |q|_2 \leq \beta \text{ a.e. in } \Omega, \tag{OC3}$$

where the (OC3) expresses that $\bar{\lambda}$ is an element of the normal cone to $-K_1^* \times K_2$ at $[\bar{z}, \bar{q}]$. Equivalently, there exists $[\bar{\mu}, \bar{\zeta}] \in Z \times L^2(\Omega)$ with

$$\Lambda A^{-1}\Lambda^*[\bar{z}, \bar{q}] - \Lambda A^{-1} l - [\psi, 0] + [\bar{\mu}, \bar{\zeta}\bar{q}] = 0, \tag{3.4}$$

$$\bar{\zeta} - \max(0, \bar{\zeta} + c(|\bar{q}|_2 - \beta)) = 0,\ c > 0, \tag{3.5}$$

$$\bar{z} \leq 0, \tag{3.6}$$

$$\langle \bar{\mu}, z^* - \bar{z} \rangle \leq 0 \ \ \forall\, z^* \leq 0. \tag{3.7}$$

In general, these conditions are not directly eligible to the semismooth Newton method in the sense of [30]: Firstly, for generalized differentiation of the mapping associated with the left hand side of (3.5) in infinite dimensions, the setting lacks a suitable norm gap, see [27, 28] and section 6. Note that these issues are absent if a direct discretization is applied which may, however, be at the cost of mesh dependent convergence rates.

Secondly, (3.7) cannot be reformulated with the help of a pointwise NCP-function, i.e., a function $\phi : \mathbb{R}^2 \to \mathbb{R}$ with the property

$$a \geq 0, b \geq 0, ab = 0 \Longleftrightarrow \phi(a, b) = 0.$$

This is due to the fact that elements of $Z^*$ in general do not allow for a pointwise interpretation. For these reasons we employ a penalization-regularization approach in the next sections.

**Interpretation of the dual variables.** Considering the second component in (P-D) and using $P^* = MP^{-1}$, we obtain a direct relation between $\bar{q}$ and the optimal stress $\bar{\sigma} := \mathbb{C}(\varepsilon(\bar{u}) - \bar{p})$:

$$P(M^{-1/2}\bar{q}) = -\bar{\sigma} + \bar{k}\bar{p} + k_1 p_0 + \mathbb{C}p_0 \quad \text{in } Q_0^*.$$

This implies

$$P(M^{-1/2}\bar{q}) = -\operatorname{dev}(\bar{\sigma} - \mathbb{C}p_0) + k_1(\bar{p} + p_0) + k_2\bar{p} \quad \text{in } Q_0, \tag{3.8}$$

which shows that $|\bar{q}|_2 - \sigma_y$ corresponds to the value of the von Mises yield function, cf. (2.11). Thus, the norm of $\bar{q}$ determines the elasto-plastic material behavior. Moreover, by multiplying (P-D) by $[u, 0], u \in V$, it may be shown, analogously to the elastic case [41, 45], that $\bar{z}$ corresponds to the normal stress $\tau_{nn}\bar{\sigma} \in Z^*$ at the contact boundary.

## 4. Regularization

In order to render the optimality conditions (OC1-3) amenable to the semismooth Newton method we now choose a Hilbert subspace $H = H_1 \times H_2 \subset L^2(\Gamma_c) \times L^2(\Omega)^d$ with dense embedding

$$H = H_1 \times H_2 \hookrightarrow Z^* \times L^2(\Omega)^d.$$

To obtain a consistent regularization, $H_1$ and $H_2$ are required to satisfy the following properties.

**Assumption 4.1** (Density of convex intersections)**.** *The following density assertions are supposed to hold:*

$$\overline{\iota_1^*(\{z \in H_1 : z \leq 0 \ \textit{a.e. on}\ \Gamma_c\})}^{Z^*} = Z_-^*,$$

$$\overline{\{q \in H_2 : |q|_2 \leq \beta \ \textit{a.e. in}\ \Omega\}}^{L^2(\Omega)^d} = \{q \in L^2(\Omega)^d : |q|_2 \leq \beta \ \textit{a.e. in}\ \Omega\},$$

*where* $Z_-^* := \{z^* \in Z^* : \langle z^*, z \rangle_{(Z^*,Z)} \leq 0 \ \ \forall\, z \geq 0\}$ *and* $\iota_1^*$ *is given by* (4.1).

We further define $L^2 := L^2(\Gamma_c) \times L^2(\Omega)^d$ and denote by

$$\iota^* = [\iota_1^*, \iota_2^*] : L^2 \hookrightarrow [Z \times L^2(\Omega)^d]^* \simeq Z^* \times L^2(\Omega)^d$$

the canonical injection

$$[z, q] \mapsto [(z, .)_{L^2(\Gamma_c)} \, |_Z, q] \; \in [Z \times L^2(\Omega)^d]^*. \tag{4.1}$$

Moreover, in the following illustration (see Figure 1) of the embedding framework including two Gelfand triples, we also specify the canonical injection

$$\tilde{\iota} : H \to L^2, [z, q] \overset{\tilde{\iota}}{\mapsto} [z, q].$$

In this section only $\iota$ and $\iota^*$ will be mentioned explicitly whereas the other injections are employed

$$\begin{tikzcd}
Z \times L^2(\Omega)^d \arrow[dr, hook, "\iota"] & & {[Z \times L^2(\Omega)^d]^*} \\
& L^2 = L^2(\Gamma_c) \times L^2(\Omega)^d \arrow[ur, hook, "\iota^*"] \arrow[dr, hook, "\tilde{\iota}^*"] & \\
H = H_1 \times H_2 \arrow[ur, hook, "\tilde{\iota}"] & & H^*
\end{tikzcd}$$

Figure 1: Gelfand triple framework for the regularization

tacitly. Suitable choices for $H_1$ and $H_2$ with regard to Assumption 4.1, possibly depending on the smoothness of $\Gamma_c$, can be made using Lemmas 5.3 and 5.4 as well as Lemma 5.5 of the subsequent section. For specific examples, we refer to section 7 below.

For algorithmic reasons it may be advantageous to include a non-negative shift parameter

$$(\hat{\mu}, \hat{\nu}) \in H_+^{1/2}(\Gamma_c) \times L_+^\infty(\Omega),$$

see [26]. Finally we replace $\beta$ by an $L^\infty$-approximation $\beta_\gamma$ which shall satisfy

$$\sigma_y \leq \beta_\gamma \leq \beta \text{ a.e.}, \quad ||\beta_\gamma - \beta||_{L^2(\Omega)} \leq \tfrac{1}{\gamma}$$

for all $\gamma$.

**Regularized problem.** Following [15] we consider the regularized problem:

$$\min J_\gamma^*(z, q) \quad \text{ over } [z, q] \in H \tag{D$_\gamma$}$$

with

$$J_\gamma^*(z, q) := F^*(\Lambda^* \iota^* [z, q]) - (z, \psi)_{L^2(\Gamma_c)} + M_\gamma^1(z) + M_\gamma^2(q) + T_\gamma(z, q),$$

where we employ the following Moreau-Yosida-type regularizations of the indicator function associated with the inequality constraints in (D):

$$M_\gamma^1(z) := \tfrac{1}{2\gamma} \|[\hat{\mu} + \gamma z]^+\|_{L^2(\Gamma_c)}^2,$$
$$M_\gamma^2(q) := \tfrac{1}{2\gamma} \|[\hat{\nu} + \gamma(|q|_2 - \beta_\gamma)]^+\|_{L^2(\Omega)}^2,$$

as well as a regularization term of Tichonov type:

$$T_\gamma([z,q]) := \tfrac{1}{2\gamma}\, b([z,q],[z,q]), \tag{4.2}$$

where $b : H \times H \to \mathbb{R}$ is a continuous and coercive bilinear form represented by the operator $B \in \mathcal{L}(H, H^*)$ with ellipticity constant $\kappa_b > 0$.

**Optimality condition.** Note that $(\mathrm{D}_\gamma)$ has a unique solution $v_\gamma = [z_\gamma, q_\gamma] \in H$ which is characterized by

$$0 = N_\gamma v_\gamma - \iota \hat{w} + ([\mu_\gamma, \nu_\gamma], \,.\,)_{L^2} \quad \text{in } H^* \tag{OC1$_\gamma$}$$

with

$$\begin{cases} \hat{w} = [\hat{z}, \hat{q}] = \Lambda A^{-1} l + [\psi, 0], \\ \mu_\gamma = [\hat{\mu} + \gamma z_\gamma]^+ \ \in L^2(\Gamma_c), \\ \nu_\gamma = [\hat{\nu} + \gamma(|q_\gamma|_2 - \beta_\gamma)]^+ \,\mathfrak{q}(q_\gamma) \in L^2(\Omega)^d, \end{cases} \tag{OC2$_\gamma$}$$

where we define $\mathfrak{q}(\,.\,) : L^2(\Omega)^d \to L^2(\Omega)^d$ by

$$\mathfrak{q}(v) := \begin{cases} \frac{v}{|v|_2} & \text{if } |v|_2 > 0, \\ 0 & \text{else.} \end{cases}$$

Furthermore, the homeomorphism $N_\gamma \in \mathcal{L}(H, H^*)$ is defined as

$$N_\gamma := \iota \Lambda A^{-1} \Lambda^* \iota^* + \tfrac{1}{\gamma} B.$$

We close this section with an important consistency result concerning $\gamma \to +\infty$. This result suggests a path-following-type approach, where the associated primal-dual-path is induced by a sequence $(\gamma_k)$ with $\gamma_k > 0$.

**Theorem 4.2** (Convergence of regularized dual solutions)**.** *Let $(\gamma) \subset \mathbb{R}^+, \gamma \to \infty$. Under Assumption* 4.1 *it holds that*

*(i)* $v_\gamma = [z_\gamma, q_\gamma] \rightharpoonup [\bar{z}, \bar{q}]$ *in* $Z^* \times L^2(\Omega)^d$,

*(ii)* $\lambda_\gamma = [\mu_\gamma, \nu_\gamma] \rightharpoonup [\bar{\mu}, \bar{\nu}]$ *in* $H_1^* \times H_2^*$,

*and*

$$\Lambda^* \iota^* v_\gamma \to \Lambda^* [\bar{z}, \bar{q}] \text{ in } Y^*.$$

Proof. See appendix Appendix A. □

As a simple consequence of the previous theorem, the sequence of approximations of the optimal displacement-strain pair and the sequence of trial stresses converge strongly to the corresponding solution of the original elasto-plastic contact problem (EPC). It may further by inferred that the sequence $(q_\gamma)$ converges even with respect to the norm topology in $L^2(\Omega)^d$.

**Corollary 4.3** (Convergence of primal solutions)**.** *Under Assumption* 4.1*, the following assertions hold true:*

*(i)* *For* $y_\gamma := A^{-1}(\Lambda^*\iota^*[z_\gamma, q_\gamma] - l)$ *it holds that* $y_\gamma \to \bar{y}$ *in* $Y$*.*

*(ii)* *For* $\sigma_\gamma := \mathbb{C}(\varepsilon(u_\gamma) - p_\gamma)$ *it holds that* $\sigma_\gamma \to \bar{\sigma}$ *in* $Q$*.*

*(iii)* *It holds that* $q_\gamma \to \bar{q}$ *in* $L^2(\Omega)^d$*.*

Proof.

(i) The statement follows from the continuity of the operator $A$.

(ii) The assertion follows from (i).

(iii) The assertion follows from (i) and the fact that $\Lambda_2^*$ is a topological isomorphism.

□

## 5. Auxiliary results on density-invariant convex intersections

In this section we discuss several conditions which lead to suitable options for the regularization space $H$ with regard to Assumption 4.1. In general, for a Banach space $V$, an arbitrary dense subset $U \subset V$ as well as a convex and closed subset $K \subset V$ the inclusion

$$K \cap U \subset K \cap V \tag{5.1}$$

is not necessarily dense even for linear subspaces $K$ and $U$. Therefore we investigate several situations relevant for our application in which the density of inclusion (5.1) is guaranteed. Readers who are merely interested in numerical aspects may as well directly consult the options for $H$ specified in section 7 and take the Assumption 4.1 for granted.

**Lemma 5.1** (intersection-invariant dense embedding)**.** *Let* $V$ *be a Hilbert space and* $U$ *a dense subset* $U \subset V$*. Let* $K \subset V$ *be nonempty, convex and closed. If the projection mapping* $P_K : V \to K$ *is* $U$*-invariant, i.e.*

$$P_K(U) \subset U,$$

*then* $\overline{U \cap K}^V = K$*, i.e.* $U \cap K$ *is dense in* $K$ *with respect to the norm in* $V$*.*

Proof. For $v \in K$ there exists a sequence $(u_n) \subset U$ with $u_n \to v$. Now, $P_K(u_n) \in U$ for all $n$ by assumption, such that

$$\|P_K(u_n) - v\|_V = \|P_K(u_n) - P_K(v)\|_V \le \|u_n - v\|_V \to 0.$$

□

**Lemma 5.2** (superposition and trace)**.** *Let* $\theta : \mathbb{R} \to \mathbb{R}$ *be Lipschitz continuous and assume that* $\theta'(t)$ *exists except for finitely many points* $t \in \mathbb{R}$*. Further let* $\Omega$ *be a Lipschitz domain. Assume that* $\mu(\Omega) < +\infty$ *or* $\theta(0) = 0$*. For the trace operator* $\tau : H^1(\Omega) \to H^{1/2}(\partial\Omega)$ *it holds that*

$$(\theta \circ \tau)(u) = (\tau \circ \theta)(u) \quad \text{in } L^2(\partial\Omega) \tag{5.2}$$

*for all* $u \in H^1(\Omega)$*.*

Proof. Under the above conditions, the superposition

$$\theta_1 = \theta : L^2(\partial\Omega) \to L^2(\partial\Omega)$$

is well-defined and continuous. Further, it is well known that the superposition

$$\theta_2 = \theta : H^1(\Omega) \to H^1(\Omega)$$

is also well-defined [34] and continuous, cf. [40]. Since (5.2) holds for any $u \in C(\overline{\Omega}) \cap H^1(\Omega)$, a density argument completes the proof. $\square$

**Lemma 5.3.** *For $L^2_-(\Gamma_c) := \{z \in L^2(\Gamma_c) : z \leq 0$ a.e. on $\Gamma_c\}$ it holds that*

$$\overline{\iota_1^*(L^2_-(\Gamma_c))}^{Z^*} = Z^*_-.$$

Proof. Define the closed, convex and nonempty set $M \subset Z^*$ by

$$M := \overline{\iota_1^*(L^2_-(\Gamma_c))}^{Z^*} \subset Z^*_-$$

and assume $Z^*_- \setminus M \neq \emptyset$. For $0 \neq z^* \in Z^*_- \setminus M$ it holds that $\alpha z^* \in Z^*_- \setminus M$ for all $\alpha > 0$. Furthermore, there exists a sequence $(v_n) \subset L^2(\Gamma_c)$ with

$$v_n \to z^* \text{ in } Z^*. \tag{5.3}$$

We first assume that $||v_n||_{L^2(\Gamma_c)} \to +\infty$ as $n \to +\infty$. The Hahn-Banach Separation Theorem implies that for all $n \in \mathbb{N}$ there exists $z_n \in Z$ with

$$\langle z_n, \tfrac{1}{||v_n||^2_{L^2(\Gamma_c)}} z^* \rangle_{(Z,Z^*)} > 1 \quad \text{and} \tag{5.4}$$

$$\langle z_n, v \rangle_{(Z,Z^*)} \leq 1 \quad \text{for all } v \in M. \tag{5.5}$$

We decompose $z_n = z_n^+ + z_n^-$ into a positive part $z_n^+ := \max(0, z)$ and a negative part $z_n^- := \min(0, z)$, where it is easy to see that $\{z_n^+, z_n^-\} \subset Z$. Indeed, recall (cf. e.g. [19, p.20]) that $Z = H^{1/2}(\Gamma_c)$ is defined by the set of all $z \in L^2(\Gamma_c)$ with finite seminorm

$$|z|^2_{\Gamma_c,1/2} := \int_{\Gamma_c} \int_{\Gamma_c} \tfrac{|z(x)-z(y)|^2}{|x-y|^n} \,\mathrm{dx}\,\mathrm{dy} < +\infty.$$

Further observe that $\max(0, z) \in L^2(\Gamma_c)$ and superposition with Lipschitz functions preserves the finiteness of the seminorm. Alternatively one may invoke Lemma 5.2. From (5.4) and $z^* \in Z^*_-$ it follows that

$$\langle z_n^-, \tfrac{1}{||v_n||^2_{L^2(\Gamma_c)}} z^* \rangle_{(Z,Z^*)} > 1,$$

in particular $z_n^- \neq 0$. Setting $v = \iota_1^*(z_n^-)||z_n^-||_Z||z_n^-||^{-2}_{L^2(\Gamma_c)}$ in (5.5), where $v \in M$ by definition, one obtains

$$\langle z_n, v \rangle_{(Z,Z^*)} = ||z_n^-||_Z \leq 1. \tag{5.6}$$

On the other hand, for $v_n$ according to (5.3) and for sufficiently large $n \in \mathbb{N}$ it holds that

$$\begin{aligned}
||z_n^-||_{L^2(\Gamma_c)} &= \sup_{v \in L^2(\Gamma_c)} \tfrac{(z_n^-, v)_{L^2(\Gamma_c)}}{||v||_{L^2(\Gamma_c)}} \\
&\geq \tfrac{(z_n^-, v_n)_{L^2(\Gamma_c)}}{||v_n||_{L^2(\Gamma_c)}} = \tfrac{\langle z_n^-, z^* \rangle_{(Z,Z^*)}}{||v_n||_{L^2(\Gamma_c)}} - \tfrac{\langle z_n^-, z^* - \iota_1^* v_n \rangle_{(Z,Z^*)}}{||v_n||_{L^2(\Gamma_c)}} \\
&\geq ||v_n||_{L^2(\Gamma_c)} - \tfrac{||z_n^-||_Z ||z^* - \iota_1^* v_n||_{Z^*}}{||v_n||_{L^2(\Gamma_c)}} \\
&\to +\infty \quad \text{for } n \to +\infty,
\end{aligned}$$

due to (5.3) and (5.6). This clearly contradicts (5.6).

If $(v_n)$ is bounded in $L^2(\Gamma_c)$, it converges weakly (along a subsequence) in $L^2(\Gamma_c)$ to an element $u \in L^2(\Gamma_c)$, such that $z^* = u$ by (5.3). This in turn implies $z^* \in L^2(\Gamma_c) \cap Z_-^* = L_-^2(\Gamma_c)$. The latter equation relies on the density of $Z_+$ in $L_+^2(\Gamma_c)$ with respect to the norm topology in $L^2(\Gamma_c)$, which holds as a consequence of Lemma 5.1. Thus it holds that $z^* \in M$, which contradicts the initial hypothesis.

□

**Lemma 5.4.** *Let $\Omega \subset \mathbb{R}^n$ be open, $\mu(\Omega) < +\infty$, $d \in \mathbb{N}$ and $\beta : \Omega \to \mathbb{R}$ measurable with $\beta(x) \geq \sigma > 0$ a.e. in $\Omega$. For*

$$K := \{u \in L^2(\Omega)^d : |u|_2 \leq \beta \text{ a.e. in } \Omega\}$$

*it holds that $K_\infty := K \cap [C_0^\infty(\Omega)]^d$ is densely contained in $K$, i.e. $\overline{K_\infty}^{L^2(\Omega)^d} = K$.*

Proof. Let $u \in K$ and $\varepsilon > 0$.

*Part I.* We first choose a function $g \in C_0^0(\Omega)^d, g = [g_1, \ldots, g_d]$, with the following properties:

$$\begin{cases} |g_j(x)| \leq |u_j(x)| & \text{a.e. in } \Omega, \\ ||g_j - u_j||_{L^2(\Omega)} & < \frac{\varepsilon}{\sqrt{d}}, \end{cases} \tag{5.7}$$

for $j = 1, \ldots, d$. A suitable choice can be made using Lusin's Theorem: In fact, there exist for all $\delta > 0$

$$K_j \subset \Omega \text{ compact }, \ \mu(\Omega \setminus K_j) < \delta, \ j = 1, \ldots, d,$$

with $u_j\,|_{K_j}$ continuous. We define the $C_0^0(\Omega)$-functions

$$g_j(x) := \tfrac{\min(\delta, \mathrm{dist}(x, \Omega \setminus K_j))}{\delta} u_j(x)$$

which fulfill (5.7) for sufficiently small $\delta$.

Therefore $g \in C_0^0(\Omega)^d$ is an element of $K$. Moreover,

$$\|g - u\|_{L^2(\Omega)^d}^2 = \sum_{j=1}^d ||g_j - u_j||_{L^2(\Omega)}^2 < \varepsilon^2.$$

We thus have shown that $K_0 := K \cap C_0^0(\Omega)^d$ is densely contained in $K$.

*Part II.* To conclude the proof we take an arbitrary sequence $(u_n)_{n\in\mathbb{N}} \subset K_0$ which fulfills

$$||u_n - u||_{L^2(\Omega)^d} \to 0.$$

Further set $\tilde{u}_n := \frac{n}{n+1} u_n \in C_0^0(\Omega)^d$.
By continuity and the hypothesis $\beta(x) \geq \sigma$ a.e. in $\Omega$ there exists $\delta_n > 0$ with

$$|\tilde{u}_n|_2(x) \leq \beta(x) - \delta_n \quad \text{for a.e. } x \in \Omega. \tag{5.8}$$

Moreover, for every $n$ a suitable mollification yields a sequence $(v_n^k)_{k\in\mathbb{N}} \subset [C_0^\infty(\Omega)]^d$ with

$$\lim_k v_n^k \to \tilde{u}_n \text{ in } C(\bar{\Omega}). \tag{5.9}$$

Combining (5.8) and the uniform convergence property (5.9) one obtains that for each $n$ there exists $k(n)$ such that $v_n^k \in K_\infty$ and $\|v_n^k - \tilde{u}_n\|_{L^2(\Omega)^d} < \frac{\varepsilon}{3}$ for all $k \geq k(n)$.

Finally choose $n$ sufficiently large such that

$$\|u - u_n\|_{L^2(\Omega)^d} < \tfrac{\varepsilon}{3}, \quad \|u_n - \tilde{u}_n\|_{L^2(\Omega)^d} < \tfrac{\varepsilon}{3}.$$

Applying the triangle inequality shows that $(v_n^{k(n)}) \subset K_\infty$ satisfies

$$\|v_n^{k(n)} - u\|_{L^2(\Omega)^d} < \varepsilon,$$

for sufficiently large $n$, which accomplishes the proof. □

**The contact boundary as a Riemannian manifold.** In order to allow for a distribution theory on the manifold $\Gamma_c$ similar to the Euclidean case, we need to define the space of test functions $C_0^\infty(\Gamma_c)$ on a manifold $\Gamma_c$ which requires a smooth structure. In connection with an associated Riemannian measure this leads to the definition of Sobolev spaces on manifolds allowing for a complete calculus theory, cf. [18]. For the alternative approach via the completion of smooth functions w.r.t. the $W^{k,p}$-norm see [24].

In the remaining part of this section we therefore assume that the contact boundary $\Gamma_c$ is smooth, i.e., a $C^\infty$-submanifold of $\mathbb{R}^n$. More precisely, since $\partial\Omega$ is assumed to have the $N^{0,1}$-property [49], $\partial\Omega$ (possibly after an appropriate orthogonal coordinate transformation) is given locally by the graph of functions $\alpha_i \in C_B^{0,1}, i = 1, \ldots, m$. We assume that those $\alpha_i$ whose graph has nonempty intersection with $\Gamma_c$, are not only in $C_B^{1,1}$ but in $C^\infty \cap C_B^{1,1}$ on an appropriate bounded domain in $\mathbb{R}^{N-1}$. Here, the space $C_B^{k,\kappa}$ is defined as the set of $k$-times continuously differentiable functions with bounded derivatives of order less than or equal $k$ and $\kappa$-Hölder-continuous $k$-th derivative [49].

In this way, $\Gamma_c$ becomes an $(N-1)$-dimensional $C^\infty$-submanifold of $\mathbb{R}^N$. We further endow the Cartesian product of the tangent spaces of $\Gamma_c$ with the usual scalar product in $\mathbb{R}^N$. This canonical construction yields a Riemannian manifold $(\Gamma_c, \langle\, .\, ,\, .\, \rangle_{\mathbb{R}^N})$.

**Lemma 5.5.** *Let $\Gamma_c$ be a $C^\infty$-submanifold of $\mathbb{R}^N$ and consider $(\Gamma_c, g)$, $g = \langle\, .\, ,\, .\, \rangle_{\mathbb{R}^N}$, as a Riemannian manifold with associated Riemannian measure $\mu = \mu(g)$. Then for $L^2_-(\Gamma_c) := \{u \in L^2(\Gamma_c) : u \leq 0 \ \mu\text{-a.e. on } \Gamma_c\}$,*

$$K_\infty := L^2_-(\Gamma_c) \cap [C_0^\infty(\Gamma_c)]$$

*is densely contained in $L^2_-(\Gamma_c)$.*

PROOF. Let $u \in L^2_-(\Gamma_c)$. Since $C_0^\infty(\Gamma_c)$ is dense in $L^2(\Gamma_c)$ [18] there exists a sequence $(v_k) \subset C_0^\infty(\Gamma_c)$, such that $v_k \to u$ in $L^2(\Gamma_c)$. We further denote by

$$\psi_k \in C^{0,1}(\mathbb{R}) \cap C^\infty(\mathbb{R}), k \in \mathbb{N},$$

non-positive functions with uniformly bounded Lipschitz modules $L_k$, i.e. $\sup_k L_k < +\infty$, which satisfy

$$\psi_k(t) \to \min(0,t) \quad \text{(pointwise)}.$$

Such a function can be easily constructed [18, Example 5.3]. Using the triangle inequality we infer

$$\begin{aligned}&||u - \psi_k(v_k)||_{L^2(\Gamma_c)} \\ &\leq \underbrace{||\min(0,u) - \psi_k(u)||_{L^2(\Gamma_c)}}_{\to 0} + \underbrace{||\psi_k(u) - \psi_k(v_k)||_{L^2(\Gamma_c)}}_{\leq L_k \|u - v_k\|_{L^2(\Gamma_c)}}\end{aligned}$$

where the convergence of the left summand follows from the Dominated Convergence Theorem. This completes the proof. □

## 6. Semismooth Newton Method

Considering the necessary and sufficient optimality conditions (OC1$_\gamma$) - (OC2$_\gamma$) of the regularized problem, the goal of this section is the application of the semismooth Newton method applied to a suitable operator equation which equivalently characterizes the optimality conditions. The notion of Newton differentiability which is applied here can be found in [11, 30] and reads as follows.

**Definition 6.1 (Newton differentiability).** Let $X, Y$ be Banach spaces and $U \subset X$ be an open set. A mapping $F : U \to Y$ is called Newton differentiable in $U$ if there exists a family of mappings $G_F : U \to \mathcal{L}(X,Y)$ which satisfy

$$\|F(x+h) - F(x) - G_F(x+h)h\|_Y = o(\|h\|_X), \quad \|h\|_X \to 0,$$

for all $x \in U$.

The corresponding generalized Newton method converges locally at a superlinear rate [11]. Further, mesh independence results [25, 29] are available. We emphasize that the semismooth Newton method has found considerable attention throughout the last decade as it has proved to be a remarkably efficient method, notably for the solution of various problems in PDE-constrained optimization [30, 26, 27] and variational inequalities [15, 28, 39], to mention only a few.

We further rely on the following calculus rules related to the Newton differentiability of several nonsmooth functions which can be found in [30] and [28].

For measurable subsets $\tilde{\Omega} \subset \Omega$ or $\tilde{\Omega} \subset \partial\Omega$ and $1 \leq q \leq p \leq \infty$, the operator $[\,.\,]^+$ defined by

$$\begin{aligned}&[\,.\,]^+ : L^p(\tilde{\Omega})^d \to L^q(\tilde{\Omega})^d, \\ &\quad v \mapsto (x \mapsto \max(0, v(x))),\end{aligned}$$

from now on always denotes the pointwise max-operator.

**Lemma 6.2** (Newton differentiability of the pointwise maximum)**.** *The pointwise maximum function* $F(\,.\,) := [\,.\,]^+$

$$F : L^p(\tilde{\Omega}) \to L^q(\tilde{\Omega}),$$

*is Newton differentiable for* $1 \le q < p \le +\infty$*. A corresponding Newton derivative is given by*

$$G_F(u)h := \begin{cases} 0, & \text{on } \mathcal{I}(u), \\ h, & \text{on } \mathcal{A}(u), \end{cases}$$

*where* $\mathcal{A}(u) := \{x \in \tilde{\Omega} : u(x) > 0\}$ *and* $\mathcal{I}(u) := \tilde{\Omega} \setminus \mathcal{A}(u)$*.*

**Lemma 6.3** (Newton differentiability of a generalized maximum function)**.** *Let* $\beta \in L^\infty(\Omega)$ *with* $\beta(x) \ge c > 0$ *a.e. in* $\Omega$*. Then the mapping*

$$\mathfrak{m} : u \mapsto [|u|_2 - \beta]^+ \mathfrak{q}(u)$$

*is Newton differentiable as a mapping from* $L^p(\Omega)^d \to L^s(\Omega)^d$ *for* $3 \le 3s \le p < +\infty$*. A corresponding Newton derivative is given by*

$$G_{\mathfrak{m}}(u) := \chi_{\mathcal{A}(u)} \cdot \mathfrak{M}(u)$$

*where*

$$\begin{cases} \rho(u) & := [|u|_2 - \beta]^+ \frac{1}{|u|_2}, \\ \mathfrak{M}(u)(\,.\,) & := \rho(u)(\,.\,) + (1 - \rho(u)) \frac{uu^\top(\,.\,)}{|u|_2^2}, \\ \mathcal{A}(u) & := \{x \in \Omega : |u|_2(x) > \beta(x)\}. \end{cases} \tag{6.1}$$

**Reformulation.** We equivalently reformulate the optimality condition (OC1$_\gamma$) for $v_\gamma$ by the nonsmooth equation

$$\Psi_\gamma(\lambda_\gamma) = 0 \tag{6.2}$$

using the operator $\Psi_\gamma : H^* \to H^*$ defined by

$$\Psi_\gamma \begin{bmatrix} \mu \\ \nu \end{bmatrix} := \begin{bmatrix} \mu \\ \nu \end{bmatrix} - \tilde{\iota}^* \begin{bmatrix} [\hat{\mu} + \gamma z(\lambda)]^+ \\ [\hat{\nu} + \gamma(|q(\lambda)|_2 - \beta_\gamma)]^+ \mathfrak{q}(q(\lambda)) \end{bmatrix},$$

where $v(\lambda) := (z(\lambda), q(\lambda)) := N_\gamma^{-1}(\iota\hat{w} - \lambda) \in H$ denotes the solution to (OC1$_\gamma$) given some candidate $\lambda$ for $\lambda_\gamma$. The superlinear convergence of the generalized Newton method

$$\lambda^{j+1} = \lambda^j - G_{\Psi_\gamma}(\lambda^j)^{-1} \Psi_\gamma(\lambda^j) \tag{6.3}$$

to solve (6.2) hinges, among others, on the Newton differentiability of $\Psi_\gamma$ in the sense of Definition 6.1. In view of the preceding calculus rules the latter relies on the following assumption.

**Assumption 6.4** (Norm gap)**.** *With regard to Lemma* 6.2 *and Lemma* 6.3*, the Newton differentiability of* $\Psi_\gamma$ *requires additional restrictions on the choice of the spaces* $H_1$ *and* $H_2$*. For this purpose, imposing the conditions*

$$H_1 \subset L^{2+\varepsilon}(\Gamma_c),\ \varepsilon > 0, \quad \text{and} \quad H_2 \subset [L^6(\Omega)]^d$$

*is sufficient.*

From now on, we assume that the regularization space $H$ is chosen in such a way that Assumption 6.4 is fulfilled. Thus, the operator $\Psi_\gamma : H^* \to H^*$ is Newton differentiable. We proceed by computing a particular Newton derivative.

Using the chain rule for Newton derivatives with affine continuous functions, we obtain the Newton derivative of $\Psi_\gamma$,

$$G_{\Psi_\gamma}(\lambda)(\,.\,) = \mathrm{id}_{H^*}(\,.\,) + \gamma\tilde{\iota}^* \begin{bmatrix} \chi_{\mathcal{Z}_\gamma(z(\lambda))} & 0 \\ 0 & \chi_{\mathcal{Q}_\gamma(q(\lambda))}\mathfrak{M}(q(\lambda)) \end{bmatrix} \circ N_\gamma^{-1}(\,.\,),$$

which includes the following quantities:

$$\begin{aligned} \rho(q) &:= [|q|_2 + \tfrac{\hat{\nu}}{\gamma} - \beta_\gamma]^+ \tfrac{1}{|q|_2}, \\ \mathfrak{M}(q(\lambda))(\,.\,) &= \rho(q(\lambda))(\,.\,) + (1 - \rho(q(\lambda)))\tfrac{q(\lambda)q(\lambda)^\top(\,.\,)}{|q(\lambda)|_2^2}, \\ \mathcal{Z}_\gamma(z) &:= \{x \in \Gamma_c : (z + \tfrac{\hat{\mu}}{\gamma})(x) > 0\}, \\ \mathcal{Q}_\gamma(q) &:= \{x \in \Omega : (|q|_2 + \tfrac{\hat{\nu}}{\gamma} - \beta_\gamma)(x) > 0\}. \end{aligned}$$

We start the analysis of the generalized Newton iteration by the following lemma.

**Lemma 6.5** (Uniform invertibility)**.** *The operator*

$$G_{\Psi_\gamma}(\lambda) \in \mathcal{L}(H^*, H^*)$$

*is uniformly invertible, i.e., for all $\delta \in H^*$ we have*

$$\|\delta\|_{H^*} \le c(\gamma)\|G_{\Psi_\gamma}(\lambda)\delta\|_{H^*}, \quad \textit{with } c(\gamma) > 0.$$

Proof. Similarly to [15] we decompose

$$G_{\Psi_\gamma}(\lambda) = \tilde{N}_\gamma(\lambda) \circ N_\gamma^{-1}$$

with

$$\tilde{N}_\gamma(\lambda) = \left(N_\gamma + \gamma\tilde{\iota}^* \begin{bmatrix} \chi_{\mathcal{Z}_\gamma(z(\lambda))} & 0 \\ 0 & \chi_{\mathcal{Q}_\gamma(q(\lambda))}\mathfrak{M}(q(\lambda)) \end{bmatrix}\right).$$

The operator $\tilde{N}_\gamma(\lambda) \in \mathcal{L}(H, H^*)$ is uniformly invertible, i.e., independently of $\lambda$, since for arbitrary $[z, q] \in H$ it holds

$$\begin{aligned} &\langle \tilde{\iota}^* \begin{bmatrix} \chi_{\mathcal{Z}_\gamma(z(\lambda))} & 0 \\ 0 & \chi_{\mathcal{Q}_\gamma(q(\lambda))}\mathfrak{M}(q(\lambda)) \end{bmatrix} \begin{bmatrix} z \\ q \end{bmatrix}, \begin{bmatrix} z \\ q \end{bmatrix} \rangle_{(H^*,H)} \\ &= (\chi_{\mathcal{Z}_\gamma(z(\lambda))} z, z)_{L^2(\Gamma_c)} + (\chi_{\mathcal{Q}_\gamma(q(\lambda))}\mathfrak{M}(q(\lambda))q, q)_{L^2(\Omega)^d} \\ &\ge \int_{Q_\gamma(q(\lambda))} \rho(q(\lambda)) \left(|q|_2^2 - \tfrac{(q(\lambda):q)^2}{|q(\lambda)|_2^2}\right) \ge 0. \end{aligned}$$

The assertion follows from the ellipticity of the bilinear form associated to $N_\gamma$. ∎

Lemma 6.5 guarantees that the iteration (6.3) and the subsequent algorithm is well-defined.

**Algorithm SSN$^\lambda(\gamma)$:** SSN algorithm in $\lambda$

**input**: $\lambda^0 := (\mu^0, \nu^0) \in H^* = H_1^* \times H_2^*$

set $j := 0$;
**while** some stopping rule is not satisfied **do**
  compute the solution $\delta_\lambda^j \in H^*$ of $G_{\Psi_\gamma}(\lambda^j)\delta_\lambda^j = -\Psi_\gamma(\lambda^j)$ ;
  set $\lambda^{j+1} := \lambda^j + \delta^j$ and $j := j+1$ ;

We immediately infer local superlinear convergence.

**Corollary 6.6** (Semismooth Newton algorithm)**.** *If $\lambda^0 \in H^*$ is sufficiently close to $\lambda_\gamma$, then the following assertions hold true:*

*(i) The Newton iterates $(\lambda^j) \subset H^*$ generated by Algorithm SSN$^\lambda(\gamma)$ converge superlinearly to $\lambda_\gamma \in L^2$.*

*(ii) The Newton iterates $(v^j) \subset H$ defined by $v^j = N_\gamma^{-1}(\iota\hat{w} - \lambda^j)$ converge superlinearly to $v_\gamma$ in $H$.*

*(iii) If $\lambda^0 \in L^2$, then $(\lambda^j)_{j\in\mathbb{N}} \subset L^2$.*

PROOF.

(i) The assertion follows directly from [30, Theorem 1.1].

(ii) The assertion is a consequence of the fact that superlinear convergence is preserved by the topological isomorphism $N_\gamma$.

(iii) If $\lambda^j \in L^2$, then we have $\Psi_\gamma(\lambda^j) \in L^2$.
The definition of the Newton step (6.3) yields

$$G_{\Psi_\gamma}(\lambda^j)\delta_\lambda^j = -\Psi_\gamma(\lambda^j) \iff$$
$$\delta_\lambda^j + \gamma \underbrace{\tilde{\iota}^* \begin{bmatrix} \chi_{\mathcal{Z}_\gamma(z(\lambda^j))} & 0 \\ 0 & \chi_{\mathcal{Q}_\gamma(q(\lambda^j))}\mathfrak{M}(q(\lambda^j)) \end{bmatrix} \circ N_\gamma^{-1}\delta_\lambda^j}_{\in L^2} = \underbrace{-\Psi_\gamma(\lambda^j)}_{\in L^2}$$

which proves the assertion.

□

Finally we specify the globalized infinite-dimensional semismooth Newton algorithm in $v$ (rather than in $\lambda$) whose local properties are analyzed in Corollary 6.6. For the globalization of our Newton-scheme one may use a line search procedure [15]. For this purpose, we need to check whether the update direction, say $\delta_v^j$ in Algorithm SSN($\gamma$), is related to the gradient of $J_\gamma^*$. This is the content of the subsequent result.

**Algorithm SSN($\gamma$):** Globalized SSN algorithm in $v$

**input**: $v^0 \in H$

set $j := 0$;
**while** some stopping rule is not satisfied **do**
  compute $\lambda^j := -N_\gamma v^j + \iota\hat{w}$ ;
  compute the solution $\delta_v^j \in H$ of $\tilde{N}_\gamma(\lambda^j)(-\delta_v^j) = -\Psi_\gamma(\lambda^j)$;
  determine $\alpha^j > 0$ by a line search method based on $\alpha \mapsto J_\gamma^*(v^j + \alpha\delta_v^j)$;
  set $v^{j+1} := v^j + \alpha^j\delta_v^j$ and $j := j + 1$ ;

**Proposition 6.7** (Gradient-relatedness)**.** *The search directions $(\delta_v^j)$ generated by Algorithm SSN($\gamma$) satisfy*

$$\langle J_\gamma^{*\prime}(v^j), \delta_v^j\rangle_{(H^*,H)} \leq -\tfrac{\kappa_b}{\gamma C(\gamma)^2}\|J_\gamma^{*\prime}(v^j)\|_{H^*}^2,$$

*where* $C(\gamma) = \sup_\lambda \|\tilde{N}_\gamma(\lambda)\| \in (0, +\infty)$.

Proof. Note that $J_\gamma^{*\prime}(v^j) = -\Psi_\gamma(\lambda^j)$. Using the definition of $\delta_v^j$ we conclude that

$$\begin{aligned}\langle J_\gamma^{*\prime}(v^j), \delta_v^j\rangle_{(H^*,H)} &= \langle J_\gamma^{*\prime}(v^j), -\tilde{N}_\gamma(\lambda^j)^{-1}(J_\gamma^{*\prime}(v^j))\rangle_{(H^*,H)} \\ &\leq -\tfrac{\kappa_b}{\gamma\|\tilde{N}_\gamma(\lambda^j)\|}\|J_\gamma^{*\prime}(v^j)\|_{H^*}^2,\end{aligned}$$

since it holds for arbitrary $g = \tilde{N}_\gamma(\lambda)v \in H^*$ that

$$\begin{aligned}\langle \tilde{N}_\gamma(\lambda)^{-1}g, g\rangle &= \langle \tilde{N}_\gamma(\lambda)v, v\rangle \\ &\geq \tfrac{\kappa_b}{\gamma}\|v\|_H^2 \geq \tfrac{\kappa_b}{\gamma}\tfrac{1}{\|\tilde{N}_\gamma(\lambda)\|^2}\|g\|_{H^*}^2.\end{aligned}$$

The definition of $\mathfrak{M}$, cf. (6.1), yields for $v = [z, q] \in H$ that

$$\begin{aligned}\|\tilde{N}_\gamma(\lambda)v\|_{H^*} &\leq \|N_\gamma v\|_{H^*} + \gamma\left\|\tilde{\iota}^*\begin{bmatrix}\chi_{\mathcal{Z}_\gamma(z(\lambda))}z \\ \chi_{\mathcal{Q}_\gamma(q(\lambda))}\mathfrak{M}(q(\lambda))q\end{bmatrix}\right\|_{H^*} \\ &\leq \|N_\gamma\|\|v\|_H + \gamma C\|v\|_{L^2} \leq (\|N_\gamma\| + \gamma C)\|v\|_H,\end{aligned}$$

where $C > 0$ may take different values on different occasions. This ends the proof. $\square$

**Remark(Global convergence).** We immediately infer that endowing the search directions $(\delta_v^j)$ with a line search method fulfilling the Armijo condition yields global convergence of the generalized Newton method [7].

## 7. Numerical Validation

In this section we validate the theoretical algorithmic framework by numerical tests. For this purpose, we specify the Tichonov regularization as well as the precise discrete setting.

**Regularization.** We propose two choices for the Tichonov regularization pair $[H, b]$.

(R1) If $\Gamma_c$ is a $C^\infty$-submanifold of $\mathbb{R}^N$, we set $H := H^1(\Gamma_c) \times H^1(\Omega)^d$, and define

$$b([z,q],[\tilde{z},\tilde{q}]) := (z,\tilde{z})_{H^1(\Gamma_c)} + (q,\tilde{q})_{H^1(\Omega)^d}.$$

(R2) Setting $H := H^{1/2}(\Gamma_c) \times H^1(\Omega)^d$, we define

$$b([z,q],[\tilde{z},\tilde{q}]) := (z,\tilde{z})_{H^{1/2}(\Gamma_c)} + (q,\tilde{q})_{H^1(\Omega)^d}.$$

Here, the $H^1-$inner product on $\Gamma_c$ is defined analogously as for the usual domain case, i.e.,

$$(z,\tilde{z})_{H^1(\Gamma_c)} := (z,\tilde{z})_{L^2(\Gamma_c)} + (\nabla z, \nabla \tilde{z})_{\vec{L}^2(\Gamma_c)},$$

where the Hilbert space $\vec{L}^2(\Gamma_c)$ is defined by the set of (equivalence classes of) vector fields $u : \Gamma_c \to T\Gamma_c$, i.e. $u(x) \in T_x\Gamma_c$ for all $x \in \Gamma_c$, with integrable Riemannian product $\langle u, u\rangle_{\mathbb{R}^N}$ on $\Gamma_c$ equipped with the canonical surface measure. Here, $T\Gamma_c := \cup_x T_x(\Gamma_c)$ denotes the tangent bundle to $\Gamma_c$. For details see section 5.

Recalling the discussion in section 5, both choices fulfill Assumption 4.1. Moreover, the Sobolev Imbedding Theorem ensures that Assumption 6.4 is satisfied [1]. Whereas (R2) is primarily of theoretical interest, alternative choices such as $H_0^1$-regularizations are also possible in view of the Poincaré-Friedrichs inequality on manifolds [46] and the results of section 5. However, due to the stress-like nature of the dual variables, cf. (3.8), we prefer not to impose additional boundary conditions.

In view of Theorem 4.2 and Corollary 4.3, Algorithm SSN($\gamma$) is embedded into an update scheme for $\gamma$, i.e. once Algorithm SSN($\gamma$) terminates successfully for a given $\gamma$, the (set of) penalty/regularization parameter(s) is increased and Algorithm SSN($\gamma$) is restarted. In order to avoid the inverse $A^{-1}$ we explicitly involve the primal variable $y$ and solve the coupled elliptic second-order system

$$\begin{bmatrix} A & -\Lambda^*\iota^* \\ \iota\Lambda & \frac{1}{\gamma}B + \tilde{\iota}^* G_M(v) \end{bmatrix} \begin{bmatrix} \delta_y \\ \delta_v \end{bmatrix} = \begin{bmatrix} 0 \\ -\iota\Lambda y + \iota[\psi, 0] - \frac{1}{\gamma}Bv - \tilde{\iota}^* M(v) \end{bmatrix}, \tag{7.1}$$

where

$$M(z,q) := \begin{bmatrix} [\hat{\mu} + \gamma z]^+ \\ [\hat{\nu} + \gamma(|q|_2 - \beta_\gamma)]^+ \mathfrak{q}(q) \end{bmatrix}, \quad G_M(z,q) := \gamma \begin{bmatrix} \chi_{\mathcal{Z}_\gamma(z)} & 0 \\ 0 & \chi_{\mathcal{Q}_\gamma(q)} \mathfrak{M}(q) \end{bmatrix}.$$

**Discretization.** In the following numerical examples $\Omega \subset \mathbb{R}^2$ is polygonal, $\Gamma_c$ is a line segment and we choose option (R1) for the Tichonov regularization. We employ a conforming finite element method to solve (7.1) numerically: let $(\mathcal{T}_h)$ be a regular triangulation of $\Omega$ with $|\mathcal{T}_h|$ elements and mesh width $h$, and $(\mathcal{S}_h)$ a partition of $\Gamma_c$ into $|\mathcal{S}_h|$ line segments with maximal length $h_c$ induced by the triangulation of $\Omega$, i.e., $\mathcal{S}_h$ is defined by those mesh nodes that lie on the contact boundary $\Gamma_c$. The discrete function spaces

$$Y_h := [P_{1,h}^{\Gamma_d}(\Omega)]^2 \times [P_{0,h}(\Omega)]^2, \quad H_{1,h} := P_{1,h}(\Gamma_c), \quad H_{2,h} := [P_{1,h}(\Omega)]^2, \tag{7.2}$$

are defined by the usual $P_0$- and $P_1-$finite element spaces

$$\begin{aligned} P_{1,h}^{\Gamma}(\Omega) &= \{u \in L^\infty(\Omega) : u|_T \in P_1 \;\forall T \in \mathcal{T}_h, u|_\Gamma = 0 \text{ a.e. }\} \cap C^0(\overline{\Omega}), \\ P_{0,h}(\Omega) &= \{u \in L^\infty(\Omega) : u|_T \in P_0 \;\forall T \in \mathcal{T}_h\}, \\ P_{1,h}(\Gamma_c) &= \{u \in L^\infty(\Gamma_c) : u|_S \in P_1 \;\forall S \in \mathcal{S}_h\} \cap C^0(\overline{\Gamma}_c), \end{aligned}$$

for $\Gamma \subset \partial\Omega$. Here $P_k$ denotes the set of polynomials of total degree less than or equal $k$ and we omit the superscript $\Gamma$ whenever $\Gamma$ has vanishing surface measure. The discretization $[P_{0,h}(\Omega)]^2$ of the space $Q_0$ is realized using the parametrization $P$ defined in (3.1). The superscript $h$ denotes the discrete version of a given linear operator corresponding to the discrete spaces (7.2).

In the discretized setting we approximate the $L^2$-norm-penalty terms in the definition of the objective in ($\mathrm{D}_\gamma$) by the standard midpoint quadrature rule and, choosing $\hat{\mu} = \hat{\nu} = 0$, one obtains the discretized-regularized problems

$$\min J^*_{\gamma,h}(z,q) \quad \text{over } [z,q] \in H_{1,h} \times H_{2,h}, \tag{$\mathrm{D}_{\gamma,h}$}$$

with

$$\begin{aligned} J^*_{\gamma,h}(z,q) := F^*_h(\Lambda^*\iota^*[z,q]) - (z,\psi)_{L^2(\Gamma_c)} + \tfrac{1}{\gamma}(z,z)_{H^1(\Gamma_c)} + \tfrac{1}{\gamma}(q,q)_{H^1(\Omega)^d} \\ + \tfrac{\gamma}{2}\sum_{k=1}^{|\mathcal{S}_h|} a_{\mathcal{S}_h,k}([\pi_{\Gamma_c} z]_k^+)^2 + \tfrac{\gamma}{2}\sum_{k=1}^{|\mathcal{T}_h|} a_{\mathcal{T}_h,k}([|[\pi_\Omega q]_k|_2 - \beta]^+)^2, \end{aligned}$$

where $a_{\mathcal{S}_h} \in \mathbb{R}^{|\mathcal{S}_h|}$ and $a_{\mathcal{T}_h} \in \mathbb{R}^{|\mathcal{T}_h|}$ denote the vectors of side lengths and element areas corresponding to the partitions $\mathcal{S}_h$ and $\mathcal{T}_h$, respectively. Employing the midpoint evaluation maps

$$\pi^h_{\Gamma_c} : H_{1,h} \to \mathbb{R}^{|\mathcal{S}_h|}, \quad \pi^h_\Omega = [\pi^h_{\Omega,1}, \pi^h_{\Omega,2}] : H_{2,h} \to \mathbb{R}^{2|\mathcal{T}_h|},$$

as well as the vectors $\mu^h_\gamma \in \mathbb{R}^{|\mathcal{S}_h|}$ and $\nu^h_\gamma \in \mathbb{R}^{2|\mathcal{T}_h|}$ given by

$$\mu^h_\gamma(z) := \gamma \operatorname{diag}(a_{\mathcal{S}_h})[\pi^h_{\Gamma_c} z]^+, \tag{7.3}$$

$$\nu^h_\gamma(q) := \operatorname{diag}(\operatorname{kron}(\begin{bmatrix}1 & 1\end{bmatrix}^\top, \zeta^h_\gamma(q)))\pi^h_\Omega(q), \tag{7.4}$$

with

$$\zeta^h_{\gamma,k}(q) := \gamma a_{\mathcal{T}_h,k}[|[\pi^h_\Omega q]_k|_2 - \beta]^+ \frac{1}{|[\pi^h_\Omega q]_k|_2}, \quad k = 1,\dots,|\mathcal{T}_h|,$$

the resulting discrete optimality system reads

$$\Psi^h_\gamma([z^h_\gamma, q^h_\gamma]) = 0, \tag{7.5}$$

where the operator $\Psi^h_\gamma : H_{1,h} \times H_{2,h} \to H^*_{1,h} \times H^*_{2,h}$ is defined by

$$\Psi^h_\gamma([z,q]) := N^h_\gamma[z,q] - \iota\hat{w} + [\pi^{h*}_{\Gamma_c}\mu^h_\gamma(z), \pi^{h*}_\Omega \nu^h_\gamma(q)].$$

Each step computation of the finite-dimensional semismooth Newton iteration applied to (7.5) requires solving the discretized version of (7.1). The discrete analogue $M^h$ to $M$ corresponding to the approximation by the midpoint quadrature rule is given by $M^h(v) = [\mu^h_\gamma(z), \nu^h_\gamma(q)]$ and its Newton derivative is denoted by $G_{M^h}$. Consequently, the resulting semismooth Newton system $[y,v] \in Y_h \times (H_{1,h} \times H_{2,h})$ reads

$$\begin{bmatrix} A^h & -\Lambda^*\iota^* \\ \iota\Lambda & \frac{1}{\gamma}B^h + G_{M^h}(v) \end{bmatrix} \begin{bmatrix} \delta_y \\ \delta_v \end{bmatrix} = \begin{bmatrix} 0 \\ -\iota\Lambda y + \iota[\psi,0] - \frac{1}{\gamma}B^h v - M^h(v) \end{bmatrix}, \tag{7.6}$$

which is posed in the space $[Y_h \times (H_{1,h} \times H_{2,h})]^*$. For a given Newton differentiable operator $\Psi_\gamma = [\Psi_{\gamma,1}, \Psi_{\gamma,2}]$ we summarize the following discrete version of Algorithm SSN($\gamma$) for fixed regularization-penalization parameter $\gamma$, mesh width $h$, starting point $v^0$ and tolerance $\varepsilon_{in}$ to solve (7.5):

---

**Algorithm SSN$(\gamma, h)$:** Globalized discrete semismooth Newton algorithm

---

**input**: $\varepsilon_{in} > 0,\ v^0 \in H_{1,h} \times H_{2,h}$

1 initialize primal variables: $y^0 \in Y_h$ by solving $A^h y^0 = \Lambda^* \iota^* v^0 - l$;
2 set $j := 0$;
3 **while** $(||\Psi^h_\gamma(v^j)||_{H^*_{1,h} \times H^*_{2,h}} < \varepsilon_{in})$ not fulfilled **do**
4     compute the solution $[\delta^j_y, \delta^j_v]$ of (7.6) in $[Y_h \times (H_{1,h} \times H_{2,h})]^*$;
5     determine $\alpha^j > 0$ by Armijo line search based on $\alpha \mapsto J^*_{\gamma,h}(v^j + \alpha \delta^j_v)$;
6     update $[y^{j+1}, v^{j+1}] := [y^j + \alpha^j \delta^j_y, v^j + \alpha^j \delta^j_v]$;
7     set $j := j + 1$;

---

The discrete norm $||\,.\,||_{H^*_{1,h} \times H^*_{2,h}}$ in step 3 of Algorithm SSN$(\gamma, h)$ is computed by solving the corresponding homogeneous coercive Neumann problems. For the implementation of the operator $A^h$ we incorporate the zero-trace condition in the definition of the space $Q_0$ using the parametrization $P$ defined in (3.1). In our numerical tests, the stopping criterion for Algorithm SSN$(\gamma, h)$ is usually set to $\varepsilon_{in} = 10^{-10}$.

**Example (a) - Screw wrench**. In this example we consider an elasto-plastic screw wrench whose geometry can be extracted from Figure 3. The elastic behavior is described by $\mathbb{C}\varepsilon = \mu_1 \operatorname{tr}(\varepsilon) I + 2\mu_2 \varepsilon$ with $\mu_1 \equiv 1.15\text{e}01$, $\mu_2 \equiv 7.69\text{e}00$. The material is assumed to satisfy the isotropic hardening law ($k_1 \equiv 0$) with $k_2 \equiv 4.0\text{e-}01$ and $\sigma_y = 2\text{e-}01$. We apply a pressure $g(x) := -6.0\text{e-}03 \cdot n(x)$ on $\Gamma_n = \operatorname{conv}(\{(5, 2.6), (8, 2)\})$. Further, we admit zero initial conditions: $\xi_0 \equiv 0, p_0 \equiv 0$ and a vanishing volume force $f \equiv 0$. Moreover, $\Gamma_d := (0, 1) \times \{2\} \cup (0, 1) \times \{3\}$, and $\Gamma_c := (0, 1) \times \{4\}$ with $\psi \equiv 1.0\text{e}00$, such that the contact constraint can be expected to be inactive at the solution and only plasticity effects have to be taken account of. The results obtained by Algorithm SSN$(\gamma, h)$ are summarized in Table 1. To verify mesh-independent convergence, we compute the solution for various fixed parameters $\gamma$ on meshes with decreasing mesh width starting from approximately $1.25 \cdot 10^4$ nodes to about $1.6 \cdot 10^6$ nodes, cf. Table 1, using uniform mesh refinement. Thereby the solution on a given mesh is prolongated to the next finer mesh to serve as a starting point $v^0$ of Algorithm SSN$(\gamma, h)$ on the refined triangulation. For validation purposes a restart strategy using the zero function as a starting point on each mesh is also tested. It is observed that the iterations count for the restart strategy stays bounded as the number of nodes are increased. Variations may be caused by the necessity for globalization in SSN$(\gamma, h)$ for higher values of $\gamma$ . The iteration numbers for the nested strategy even tend to decrease with decreasing mesh width. The theoretical property of local mesh-independent superlinear rate of convergence is verified experimentally by investigating the convergence quotients $Q_j$ associated with the iterates $(v^j)$ generated for fixed $(\gamma, h)$,

$$Q_j := \frac{||v^{(\omega-5+j)} - v^\star||_H}{||v^{(\omega-6+j)} - v^\star||_H}, \quad j = 1 \ldots, 5,$$

where $\omega$ denotes the iteration count for Algorithm SSN$(\gamma, h)$ and $v^\star$ denotes the solution obtained by applying the same algorithm with higher precision $\varepsilon_{in} = 10^{-14}$. As predicted by the theory, the convergence quotients $Q_j$ tend to zero and rest stable under decreasing mesh width even for large $\gamma$, cf. Figure 4. This clearly indicates mesh-independent convergence behavior for each fixed $\gamma$. Applying the heuristic inexact path-following approach IPF$(h)$ with regard to the penalty parameter set $\gamma$ (cf. below), we display in Figure 3 the resulting approximate optimal plastic strain $\tilde{p}$ as well the regions of extensive plastic straining on the deformed configuration. Employing relation (3.8), we also plot the approximate yield function, see Figure 2.

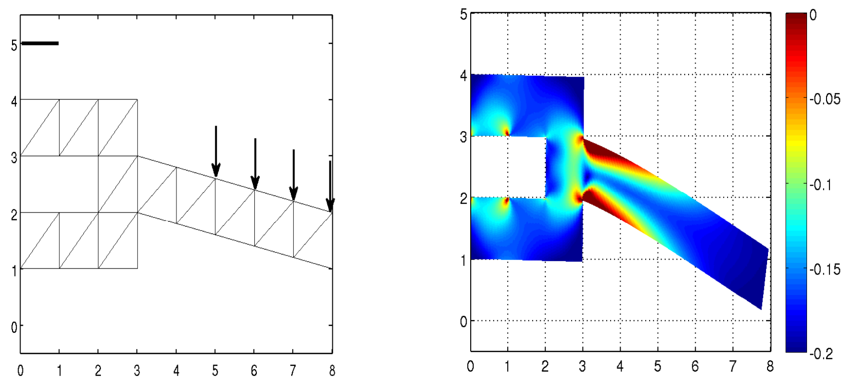


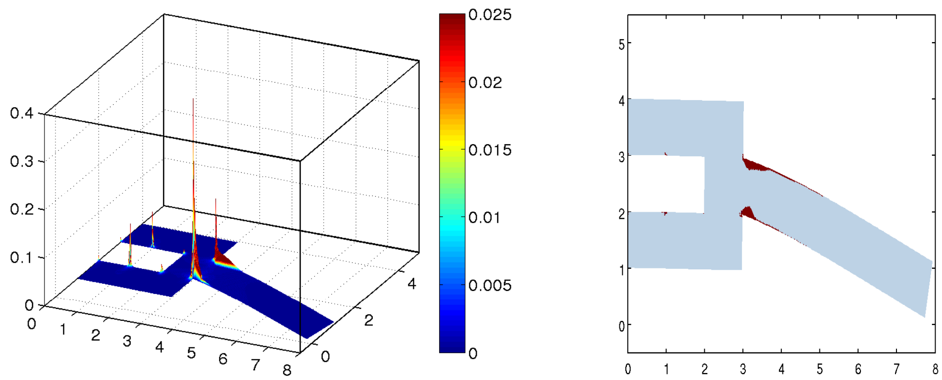
Figure 2: Example (a): initial configuration *(left)*, yield functional *(right)*

Table 1: Algorithm SSN$(\gamma, h)$, Example (a), $\varepsilon_{in}$=1.0e-10: no. of iterations w.r.t. mesh size and $\gamma$, * fixed starting point

| $\gamma$ / # nodes | 12,5k | 25k | 50k | 100k | 200k | 400k | 800k | 1.6M |
|---|---|---|---|---|---|---|---|---|
| 1.0e01 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 1.0e02 | 4 | 4 | 3 | 3 | 2 | 2 | 2 | 2 |
| 1.0e03 | 7 | 7 | 6 | 5 | 5 | 4 | 5 | 3 |
| 1.0e04 | 32 | 18 | 21 | 16 | 15 | 11 | 10 | 9 |
| 1.0e04* | 22 | 29 | 28 | 22 | 22 | 24 | 22 | 24 |
| 1.0e05 | 79 | 64 | 54 | 66 | 67 | 60 | 51 | 30 |
| 1.0e05* | 62 | 66 | 61 | 57 | 63 | 71 | 63 | 58 |

**Example (b) - L-shape.** We consider an L-shaped domain $\Omega = (0, 0.5] \times (0.5, 1) \cup (0.5, 1) \times (0, 1)$ and assume that the elastic behavior of the material is described by $\mathbb{C}\varepsilon = \mu_1 \operatorname{tr}(\varepsilon)I + 2\mu_2\varepsilon$ with $\mu_1 = \mu_2 \equiv 1.0\text{e}03$. It is further assumed that the material obeys the kinematic hardening law, i.e. $k_2 \equiv 0$. The plastic material parameters are given as follows: $\sigma_y = 2.0\text{e}01$, $k_1 \equiv 100$. The body shall be fixed at $\Gamma_d = (0.5, 1) \times \{0\}$. We set $\psi \equiv 4.0\text{e-}02$ on $\Gamma_c = (0, 1) \times \{1\}$ and apply a pressure $g(x) = -2.0\text{e}01 \cdot n(x)$ on $\Gamma_n = (0, 0.5) \times \{0.5\}$ which leads to a nonempty contact region at the solution. We further admit zero initial conditions: $\xi_0 \equiv 0, p_0 \equiv 0$ and vanishing volume force $f \equiv 0$. To verify mesh-independent convergence of Algorithm SSN$(\gamma, h)$, we compute the solution for each fixed $\gamma$ on meshes with decreasing mesh width, cf. Table 2, using uniform mesh refinement as in Example (a). Again, as starting point for each mesh we choose the (prolongated) solution of the

Figure 3: Example (a): plastic strain $|\tilde{p}|_F$ *(left)*, dominant plastic zones *(dark)*, i.e. $|\tilde{p}|_F >$ 1e-02 *(right)*



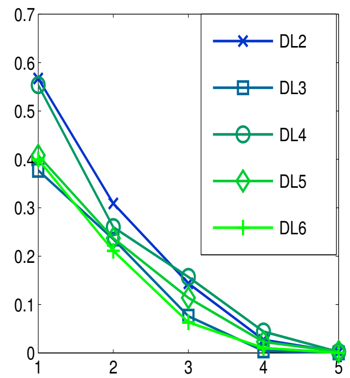



Figure 4: Example (a): $Q_j, j = 1, \ldots, 5$, for $\gamma = 1.0e05$ and various discretization levels (DL)

preceding coarser mesh. It is observed that below $\gamma \approx 1.0e04$, both active set approximations of contact and plasticity constraints are empty. For $\gamma$ between 1.0e04 and 1.0e05, only the contact constraint has a nonempty active set. Starting from $\gamma \approx$ 1.0e05 both, plastic and contact, effects need to be dealt with. Considering Table 2 we observe that the number of iterations even tends to decrease with smaller mesh width. This clearly indicates mesh-independent convergence behavior for fixed $\gamma$ as the mesh width tends to zero.

As the result of the application of the inexact path-following approach IPF$(h)$ with regard to the penalty parameter set $\gamma$ (cf. below), we display in Figure 5 the approximate optimal plastic strain as well as the regions of extensive plastic straining in the deformed configuration. Employing relation (3.8), we also plot the approximate yield function in the deformed configuration and the normal stress component on the initial configuration in Figure 6.

**Inexact Path-Following.** In order to study convergence with regard to the regularization-penalization-parameter $\gamma$ we implement a heuristic version of the inexact path-following (IPF) approach designed for the obstacle problem [26]. In contrast to the aforegoing sections, we assume that the penalization-regularization parameters are not equal, that is, we assume $\gamma = [\gamma_1, \gamma_2, \gamma_3, \gamma_4] \in \mathbb{R}^4_+$

Table 2: Algorithm SSN($\gamma, h$), Example (b), $\varepsilon_{in}$=1.0e-10: no. of iterations w.r.t. mesh size and fixed $\gamma$, $^*$ for vector-valued $\gamma$ cf. (7.7)

| $\gamma$ / # nodes | 1.6k | 6k | 25k | 100k | 400k | 1.6M |
|---|---|---|---|---|---|---|
| 1.0e03 | 1 | 1 | 1 | 1 | 1 | 1 |
| 1.0e04 | 4 | 4 | 3 | 3 | 1 | 1 |
| 5.0e04 | 4 | 9 | 8 | 4 | 3 | 5 |
| 1.0e05 | 22 | 24 | 25 | 16 | 13 | 9 |
| [1.0e06, 1.0e06, 1.0e03, 1.0e03]$^*$ | 42 | 41 | 27 | 20 | 13 | 10 |

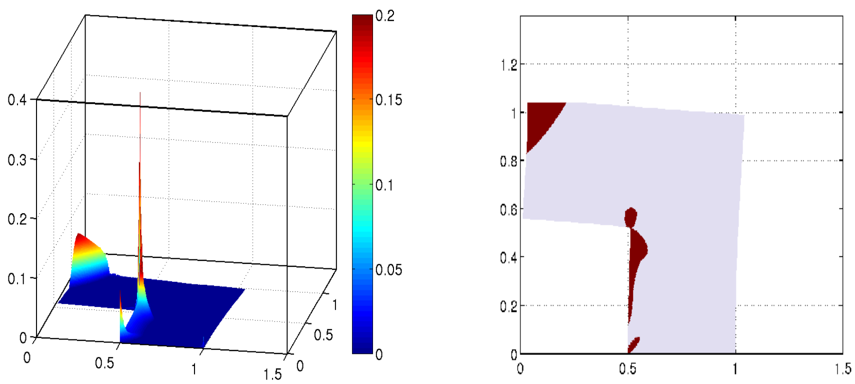


Figure 5: Example (b): $|\tilde{p}|_F$ *(left)*, dominant plastic zones *(dark)*, i.e. $|\tilde{p}|_F > 0.1$ *(right)*

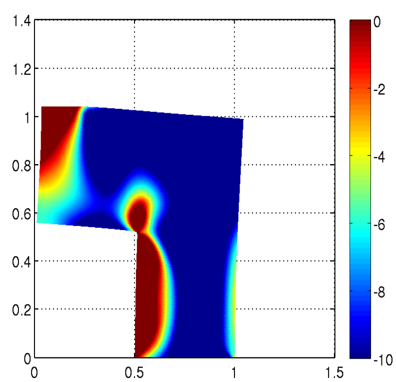


Figure 6: Example (b): yield functional

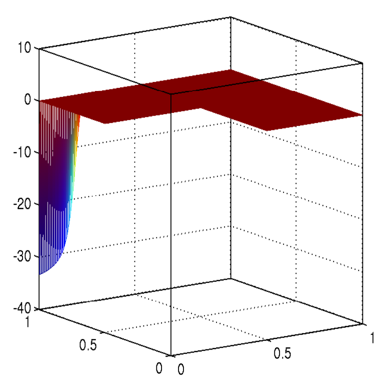


Figure 7: Example (b): normal stress approximation $\tau_{nn}\bar{\sigma}$ on $\Gamma_c$

and that the objective functional in $(\mathrm{D}_{\gamma,h})$ is given by

$$J^*_{\gamma,h}(z,q) := F^*_h(\Lambda^*\iota^*[z,q]) - (z,\psi)_{L^2(\Gamma_c)} + \tfrac{1}{\gamma_1}(z,z)_{H^1(\Gamma_c)} + \tfrac{1}{\gamma_2}(q,q)_{H^1(\Omega)^d}$$
$$+ \tfrac{\gamma_3}{2}\sum_{k=1}^{|\mathcal{S}_h|} a_{\mathcal{S}_h,k}([\pi_{\Gamma_c} z]_k^+)^2 + \tfrac{\gamma_4}{2}\sum_{k=1}^{|\mathcal{T}_h|} a_{\mathcal{T}_h,k}([|[\pi_\Omega q]_k|_2 - \beta]^+)^2. \qquad (7.7)$$

Starting from a componentwise positive parameter set $\gamma_k = [\gamma_{k,1}, \gamma_{k,2}, \gamma_{k,3}, \gamma_{k,4}] \in \mathbb{R}^4_+$, each subproblem $(D_{\gamma_k,h})$ is only solved approximately with increasing precision using Algorithm SSN$(\gamma, h)$ with $\gamma := \gamma_k$ using the modified stopping criterion

$$(||\Psi^h_{\gamma_k,1}(\tilde{v})||_{H^*_{1,h}} < \tfrac{\tau_{in}}{\gamma_1}) \wedge (||\Psi^h_{\gamma_k,2}(\tilde{v})||_{H^*_{2,h}} < \tfrac{\tau_{in}}{\gamma_2}), \quad \tau_{in} > 0, \tag{7.8}$$

which replaces line 3 of SSN$(\gamma, h)$. After a suitable update of the parameter set $\gamma_k$ which is based on the individual residuals (line 6), the computed approximate solution $\tilde{v} \approx v^h_{\gamma_k}$ is used as a starting point for the solution of the subsequent problem $(D_{\gamma_{k+1},h})$. In this way the effort of approximatively solving the subproblems can be expected to be kept rather low. Differently from [26] we are testing a constant augmentation of the $(\gamma_k)$ driven by a factor $\theta > 0$. For the outer stopping criterion we consider the optimality conditions for the solution $[z^h, q^h]$ of the discrete limit problem:

$$\begin{aligned} \Psi^h(z^h, q^h) := \iota\Lambda {A^h}^{-1}\Lambda^*\iota^*[z^h, q^h] - \iota\hat{w} + [\pi^{h*}_{\Gamma_c}\mu^h, \pi^{h*}_{\Omega}\nu^h] &= 0 \quad \text{in } H^*_{1,h} \times H^*_{2,h}, \\ \mu^h - \max(0, \mu^h + \pi^h_{\Gamma_c} z^h) &= 0 \quad \text{in } \mathbb{R}^{|\mathcal{S}_h|}, \\ \zeta^h - \max(0, \zeta^h + |\pi^h_\Omega q^h|_2 - \beta) &= 0 \quad \text{in } \mathbb{R}^{|\mathcal{T}_h|}, \end{aligned}$$

with $\nu^h = \mathrm{diag}(\mathrm{kron}(\begin{bmatrix}1 & 1\end{bmatrix}^\top, \zeta^h))\pi^h_\Omega(q^h)$. We define the associated residuals $r_{h,i}, i = 1, .., 4$, for given iterates $[z, q]$ and associated multipliers $[\mu(z), \nu(z)]$ by

$$\begin{aligned} r_{h,1}(z, q) &:= ||\Psi^h_1(z, q)||_{H^*_{1,h}}, \\ r_{h,2}(z, q) &:= ||\Psi^h_2(z, q)||_{H^*_{2,h}}, \\ r_{h,3}(z, q) &:= ||\mu(z) - \max(0, \mu(z) + \pi^h_{\Gamma_c} z)||_{L^2_h(\Gamma_C)}, \\ r_{h,4}(z, q) &:= ||\zeta(q) - \max(0, \zeta(q) + |\pi^h_\Omega q^h|_2 - \beta)||_{L^2_h(\Omega)}, \end{aligned}$$

where $||\,.\,||_{L^2_h(\,.\,)}$ denotes the $L^2$-norm of the corresponding piecewise constant midpoint interpolate. In Step 2 of Algorithm IPF$(h)$, the Lagrange multiplier candidates for $\mu, \nu$ are chosen as $\mu^h_\gamma(\tilde{z}^k)$ and $\nu^h_\gamma(\tilde{q}^k)$ which have been defined in (7.3) and (7.4).

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**Algorithm IPF$(h)$:** Inexact path-following algorithm

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**input**: $\gamma_0 \in \mathbb{R}^4_+, \theta > 1, \tau_{in} > 0, \varepsilon_{out} > 0,\ \tilde{v}^0 = [\tilde{z}^0, \tilde{q}^0] \in H_{1,h} \times H_{2,h}$

```
1 set k := 0;
2 while (|r_h(ṽ^k)| < ε_out) not fulfilled do
3     apply Algorithm SSN(γ, h) with γ = γ_k, v^0 = ṽ^k to find ṽ ∈ H_{1,h} × H_{2,h} satisfying (7.8) ;
4     for i = 1, ..., 4 do
5         if r_{h,i}(ṽ) > ε_out then
6             γ_{k+1,i} := γ_{k,i} · θ;
7     update ṽ^{k+1} := ṽ ;
8     set k := k + 1;
```

---

Tables 3 and 4 show the results for the application of Algorithm IPF$(h)$ to Examples (a) and (b), respectively, for fixed outer stopping criterion $\varepsilon_{out} = 10^{-5}$. For validation purposes we first test IPF($h$) on various meshes using for each mesh the solution obtained by SSN($\gamma, h$) with $\gamma = \gamma_0$ as a starting point $\tilde{v}_0$. This restart strategy is observed to converge mesh-independently. To keep high-dimensional calculations as low as possible we also test a nested iteration. In this approach the solution on a given mesh is prolongated to the next finer mesh to serve as a starting point $v^0$ of Algorithm IPF($h$) on the refined mesh together with the final parameter set of the coarser mesh. In this way, the major part of the computations related to the identification of the appropriate parameter set is transferred to the smallest mesh. With this strategy, no further $\gamma$-updates are necessary after the computation on the coarsest mesh and the total number of inner (SSN($\gamma, h$)) iterations decreases significantly as the number of nodes increases.

It should be pointed out that a straightforward application of Algorithm SSN($\gamma, h$) to $(D_{\gamma_{\text{end}}})$, where $\gamma_{end}$ denotes the final parameter set, replacing the stopping criterion by the respective inexact version (7.8) used in line 3 of Algorithm IPF($h$) typically requires a multiple of the iterations which shows the advantage of our path-following approach.

| # nodes | 3k | 6k | 12,5k | 25k | 50k | 100k | 200k | 400k | 800k | 1.6M |
|---|---|---|---|---|---|---|---|---|---|---|
| restart | 9(30) | 9(44) | 9(40) | 9(38) | 9(40) | 9(41) | 9(32) | 9(30) | 9(30) | 9(26) |
| nested | 9(30) | 1(23) | 1(21) | 1(13) | 1(23) | 1(19) | 1(14) | 1(15) | 1(14) | 1(9) |

Table 3: No. of outer(total inner) iterations IPF($h$), $\gamma_0 = 1.0\text{e}03 \cdot [1, 1, 1, 1]$, $\varepsilon_{out} = 1.0\text{e-}05$, $\theta = 2$ and $\tau_{in} = 1.0\text{e}00$ for Example (a)

| # nodes | 1.5k | 6k | 25k | 100k | 400k | 1.6M |
|---|---|---|---|---|---|---|
| restart | 5(136) | 5(178) | 5(151) | 5(136) | 5(127) | 5(124) |
| nested | 5(136) | 1(85) | 1(67) | 1(50) | 1(47) | 1(24) |

Table 4: No. of outer(total inner) iterations for IPF($h$), $\gamma_0 = [1.0e06, 1.0e06, 1.0e03, 1.0e03]$, $\varepsilon_{out} = 1.0\text{e-}05$, $\theta = 2$ and $\tau_{in} = 1.0\text{e}00$ for Example (b)

**Outlook.** A suitable path-following strategy leading to an automated regularization-discretization update procedure promises a higher efficiency compared to the heuristic used in Algorithm IPF($h$). For variational inequalities of the first kind these methods are already well established and prove to be remarkably efficient; see e.g. [26]. In this regard alternative choices for the coupling of the parameter $\gamma$ for both Moreau-Yosida regularizations and the Tichonov regularization may be preferable. In view of the singularities of the solutions corresponding to Examples (a) and (b), usage of adaptive strategies is strongly recommended. It should be pointed out that the approach presented in this paper can be extended to contact problems with Tresca friction. These problems are characterized by an additional weighted $L^1(\Gamma_c)$-norm functional resulting in an additional inequality in the dual problem.

## Appendix A. Proof of Theorem 4.2

Proof. Throughout the proof we use $K > 0$ as a constant which may take different values on different occasions.

**Step 1:** $(q_\gamma) \subset L^2(\Omega)^d$ is bounded.

Multiplying (OC1$_\gamma$) by $v_\gamma = [z_\gamma, q_\gamma]$ yields

$$\langle \Lambda^* \iota^* v_\gamma, A^{-1} \Lambda^* \iota^* v_\gamma \rangle_{(Y^*, Y)} + \frac{1}{\gamma} b(v_\gamma, v_\gamma) - (\hat{w}, v_\gamma)_{L^2} + (\lambda_\gamma, v_\gamma)_{L^2} = 0.$$

This induces

$$\langle \Lambda^* \iota^* v_\gamma, A^{-1}\Lambda^*\iota^* v_\gamma\rangle_{(Y^*,Y)} + (\lambda_\gamma, v_\gamma)_{L^2}, \leq \|\hat{q}\|_{L^2(\Omega)^d}\|q_\gamma\|_{L^2(\Omega)^d} + \iota_1^*(z_\gamma)(\hat{z}),$$

and thus

$$\kappa\|\Lambda^*\iota^* v_\gamma\|^2_{Y^*} + (\mu_\gamma, z_\gamma)_{L^2(\Gamma_c)} + (\nu_\gamma, q_\gamma)_{L^2(\Omega)^d} \leq K(\|z_\gamma\|_{Z^*} + \|q_\gamma\|_{L^2(\Omega)^d}). \tag{A.1}$$

Using the domain decomposition approach for $\Omega$ from [15] we get

$$\begin{aligned}\|q_\gamma\|^2_{L^2(\Omega)^d} + \tfrac{1}{\gamma}\kappa\|\Lambda^*\iota^* v_\gamma\|^2_{Y^*} + \tfrac{1}{\gamma}(\mu_\gamma, z_\gamma)_{L^2(\Gamma_c)}& \\ \leq \tfrac{K}{\gamma}(\|z_\gamma\|_{Z^*} + \|q_\gamma\|_{L^2(\Omega)^d}) &+ K\|q_\gamma\|_{L^2(\Omega)^d},\end{aligned}$$

which implies

$$\|q_\gamma\|^2_{L^2(\Omega)^d} + \tfrac{\kappa}{\gamma\|\Lambda_1^{-*}\|}\|z_\gamma\|^2_{Z^*} + \tfrac{1}{\gamma}(\mu_\gamma, z_\gamma)_{L^2(\Gamma_c)} \leq \tfrac{K}{\gamma}\|z_\gamma\|_{Z^*} + K\|q_\gamma\|_{L^2(\Omega)^d}. \tag{A.2}$$

Next we consider the boundary term $(\mu_\gamma, z_\gamma)_{L^2(\Gamma_c)}$:

$$\begin{aligned}(\mu_\gamma, z_\gamma)_{L^2(\Gamma_c)} &= (\mu_\gamma, \tfrac{1}{\gamma}\hat{\mu} + z_\gamma - \tfrac{1}{\gamma}\hat{\mu})_{L^2(\Gamma_c)} \\ &= \tfrac{1}{\gamma}\|\mu_\gamma\|^2_{L^2(\Gamma_c)} - \tfrac{1}{\gamma}(\mu_\gamma, \hat{\mu})_{L^2(\Gamma_c)} \\ &= \tfrac{1}{2\gamma}\|\mu_\gamma - \hat{\mu}\|^2_{L^2(\Gamma_c)} + \tfrac{1}{2\gamma}\|\mu_\gamma\|^2_{L^2(\Gamma_c)} - \tfrac{1}{2\gamma}\|\hat{\mu}\|^2_{L^2(\Gamma_c)} \\ &\geq -\tfrac{1}{2\gamma}\|\hat{\mu}\|^2_{L^2(\Gamma_c)}.\end{aligned} \tag{A.3}$$

Consequently we obtain from (A.2)

$$\|q_\gamma\|^2_{L^2(\Omega)^d} + \tfrac{1}{\gamma K}\|z_\gamma\|^2_{Z^*} \leq K + \tfrac{K}{\gamma}(\|z_\gamma\|_{Z^*}) + K\|q_\gamma\|_{L^2(\Omega)^d},$$

from which we conclude that $(q_\gamma)$ is bounded in $L^2(\Omega)^d$, and $(\frac{1}{\sqrt{\gamma}}z_\gamma)$ is bounded in $Z^*$.

**Step 2**: $(z_\gamma) \subset Z^*$ is bounded.

Reconsidering (A.1) we have

$$\kappa\|\Lambda^*\iota^* v_\gamma\|^2_{Y^*} + (\mu_\gamma, z_\gamma)_{L^2(\Gamma_c)} + \underbrace{(\nu_\gamma, q_\gamma)_{L^2(\Omega)^d}}_{\geq 0} \leq K(\|z_\gamma\|_{Z^*} + \|q_\gamma\|_{L^2(\Omega)^d}).$$

Similarly to the above estimates, we get

$$\tfrac{1}{K}\|z_\gamma\|^2_{Z^*} \leq K + K\|z_\gamma\|_{Z^*} + K\|q_\gamma\|_{L^2(\Omega)^d},$$

which yields that $(z_\gamma)$ is bounded in $Z^*$ according to step 1.

**Step 3**: $[\mu_\gamma, \nu_\gamma] \subset H_1^* \times H_2^*$ is bounded.

Using the results from step 1, we again multiply (OC1$_\gamma$) by $v_\gamma$ to obtain

$$\tfrac{1}{\gamma}b(v_\gamma, v_\gamma) + \underbrace{([\mu_\gamma, \nu_\gamma], v_\gamma)_{L^2}}_{\geq -K} \leq \underbrace{K\|q_\gamma\|_{L^2(\Omega)^d} + K\|z_\gamma\|_{Z^*}}_{\leq K},$$

and thus

$$\|\tfrac{1}{\sqrt{\gamma}}v_\gamma\|_H \le K. \tag{A.4}$$

Taking the $\|.\|_{H^*}$-norm in (OC1$_\gamma$) yields

$$\begin{aligned}\|[\mu_\gamma,\nu_\gamma]\|_{H^*} &\le \|\iota\hat{w}\|_{H^*} + \|\iota\Lambda A^{-1}\Lambda^*\iota^* v_\gamma\|_{H^*} + \tfrac{1}{\gamma}\|Bv_\gamma\|_{H^*}\\ &\le K + \|\iota\Lambda A^{-1}\Lambda^*\|\|\iota^* v_\gamma\|_{Z^*\times L^2(\Omega)^d} + \tfrac{1}{\gamma}\|B\|\|v_\gamma\|_H.\end{aligned}$$

Taking account of step 1 and 2 as well as (A.4), this proves the assertion.

We thus have

$$[z_\gamma,q_\gamma,\mu_\gamma,\nu_\gamma] \rightharpoonup [\tilde z,\tilde q,\tilde\mu,\tilde\nu] \in Z^*\times L^2(\Omega)^d\times H_1^*\times H_2^*,$$

for an appropriate subsequence $[z_\gamma,q_\gamma,\mu_\gamma,\nu_\gamma]\subset H_1\times H_2\times L^2(\Gamma_c)\times L^2(\Omega)^d$, sharing the same indices by abuse of notation.

**Step 4**: $\tilde v := [\tilde z,\tilde q]$ is feasible, i.e. (OC2) holds.

With step 1 and 2 it is easily seen that

$$F^*(\Lambda^*\iota^*[z_\gamma,q_\gamma]) + \underbrace{T_\gamma(q_\gamma)}_{\ge 0}$$

is bounded from below. Moreover, we have

$$\begin{aligned}J_\gamma^*(z_\gamma,q_\gamma) &\le J_\gamma^*(z,q)\\ &= F^*(\Lambda^*\iota^*[z,q]) + \tfrac{1}{2\gamma}b([z,q],[z,q]) < K\end{aligned}$$

for all $[z,q]\in Z\times L^2(\Omega)^d$ with $z\le -\frac{\hat\mu}{\gamma}$ a.e. in $\Gamma_c$, and $|q|_2\le \sigma_y-\frac{\hat\nu}{\gamma}$ a.e. in $\Omega$, with $\gamma$ sufficiently large. Consequently, $M_\gamma^1(z_\gamma)+M_\gamma^2(q_\gamma)$ is bounded.

In a similar fashion as in [15], we exploit the weak lower semicontinuity of

$$L^2(\Omega)^d \ni q\mapsto \|[|q|_2-\beta]^+\|^2_{L^2(\Gamma_c)}\in\mathbb{R}$$

to conclude that

$$\begin{aligned}0 = \int_\Omega \left([|\tilde q|_2-\beta]^+\right)^2\mathrm{dx} &\le \liminf_\gamma \int_\Omega \left([|q_\gamma|_2-\beta]^+\right)^2\mathrm{dx}\\ &\le \liminf_\gamma \int_\Omega \left([|q_\gamma|_2-(\beta_\gamma-\tfrac{\hat\nu}{\gamma})]^+\right)^2\mathrm{dx}\longrightarrow 0,\end{aligned}$$

since $M_\gamma^2(q_\gamma)=\frac{\gamma}{2}\|[|q_\gamma|_2-(\beta_\gamma-\frac{\hat\nu}{\gamma})]^+\|^2_{L^2(\Omega)}$ is bounded.

Furthermore for $z\in Z, z\ge 0$, we obtain

$$\begin{aligned}\langle[z_\gamma+\tfrac{\hat\mu}{\gamma}]^+,z\rangle_{(Z^*,Z)} &= ([z_\gamma+\tfrac{\hat\mu}{\gamma}]^+,z)_{L^2(\Gamma_c)}\\ &\ge (z_\gamma,z)_{L^2(\Gamma_c)}+\tfrac{1}{\gamma}(\hat\mu,z)_{L^2(\Gamma_c)}\to\langle\tilde z,z\rangle_{(Z^*,Z)}.\end{aligned}$$

On the other hand

$$\begin{aligned}\langle [z_\gamma + \tfrac{\hat{\mu}}{\gamma}]^+, z\rangle_{(Z^*,Z)} &\le \|[z_\gamma + \tfrac{\hat{\mu}}{\gamma}]^+\|_{Z^*}\|z\|_Z \\ &\le K\|[z_\gamma + \tfrac{\hat{\mu}}{\gamma}]^+\|_{L^2(\Gamma_c)}\|z\|_Z \to 0,\end{aligned}$$

as $L^2(\Gamma_c) \overset{\iota_1^*}{\hookrightarrow} Z^*$ and by the boundedness of $M^1_\gamma(z_\gamma)$. This accomplishes step 4.

**Step 5**: (OC1) is satisfied.

For $v \in H$, (OC1$_\gamma$) reads

$$\begin{aligned}0 &= \langle \iota^* v_\gamma, \Lambda A^{-1}\Lambda^*\iota^* v\rangle_{(Z^*\times L^2(\Omega)^d, Z\times L^2(\Omega)^d)} + \tfrac{1}{\gamma} b(v_\gamma, v) \\ &\quad - (\iota\hat{w}, v)_{L^2} + ([\mu_\gamma, \nu_\gamma], v)_{L^2}.\end{aligned}$$

Passing to the limit as $\gamma \to +\infty$ yields for $v \in H$

$$\begin{aligned}0 &= \langle \Lambda A^{-1}\Lambda^*[\tilde{z},\tilde{q}], \iota^* v\rangle_{(Z\times L^2(\Omega)^d, Z^*\times L^2(\Omega)^d)} - (\iota\hat{w}, v)_{L^2} + \langle [\tilde{\mu},\tilde{\nu}], v\rangle_{(H^*,H)} \\ &= (\Lambda A^{-1}\Lambda^*[\tilde{z},\tilde{q}], v)_{L^2} - (\iota\hat{w}, v)_{L^2} + \langle [\tilde{\mu},\tilde{\nu}], v\rangle_{(H^*,H)}.\end{aligned}$$

From the density of $H$ in $L^2$ we infer

$$-[\tilde{\mu},\tilde{\nu}] = \Lambda A^{-1}\Lambda^*[\tilde{z},\tilde{q}] - \Lambda A^{-1} l - [\psi, 0],$$

and thus (OC1).

**Step 6**: It holds that $\Lambda^*\iota^* v_\gamma \to \Lambda^*[\tilde{z},\tilde{q}]$ in $Y^*$.

By the weak lower semicontinuity of $F^*(\Lambda^* .\,)$ we have

$$\liminf_{\gamma\to+\infty} F^*(\Lambda^*\iota^*[z_\gamma, q_\gamma]) \ge F^*(\Lambda^*[\tilde{z},\tilde{q}]).$$

On the other hand, exploiting the minimality of $[z_\gamma, q_\gamma]$, we obtain for any $[z,q] \in H$ with $z \le 0, |q|_2 \le \beta$,

$$\begin{aligned}F^*(\Lambda^*\iota^*[z_\gamma, q_\gamma]) &\le F^*(\Lambda^*\iota^*[z,q]) + \tfrac{1}{2\gamma}\|[\hat{\mu} + \gamma z]^+\|^2_{L^2(\Gamma_c)} \\ &\quad + \tfrac{1}{2\gamma}\|[\hat{\nu} + \gamma(|q|_2 - \beta_\gamma)]^+\|^2_{L^2(\Omega)} + \tfrac{1}{2\gamma} b([z,q],[z,q]),\end{aligned}$$

such that

$$\begin{aligned}&\limsup_{\gamma\to+\infty} F^*(\Lambda^*\iota^*[z_\gamma, q_\gamma]) \\ &\le F^*(\Lambda^*\iota^*[z,q]) + \limsup_{\gamma\to+\infty} \tfrac{1}{2\gamma}\|\hat{\mu}\|^2_{L^2(\Gamma_c)} + \limsup_{\gamma\to+\infty} \tfrac{1}{2\gamma}\|\hat{\nu} + \gamma(\beta - \beta_\gamma)]^+\|^2_{L^2(\Omega)} \\ &= F^*(\Lambda^*\iota^*[z,q]) + \limsup_{\gamma\to+\infty} \tfrac{1}{2\gamma}\|\hat{\nu}\|^2_{L^2(\Omega)} + \limsup_{\gamma\to+\infty} (\hat{\nu}, \beta - \beta_\gamma)_{L^2(\Omega)} + \limsup_{\gamma\to+\infty} \tfrac{\gamma}{2}\|(\beta - \beta_\gamma)\|^2_{L^2(\Omega)} \\ &= F^*(\Lambda^*\iota^*[z,q]),\end{aligned}$$

since $\|\beta - \beta_\gamma\| \le \frac{1}{\gamma}$. By Assumption 4.1 we conclude

$$\limsup_{\gamma\to+\infty} F^*(\Lambda^*\iota^*[z_\gamma, q_\gamma]) \le F^*(\Lambda^*[\tilde{z},\tilde{q}]),$$

and thus

$$\lim_{\gamma\to+\infty} F^*(\Lambda^*\iota^*[z_\gamma,q_\gamma]) = F^*(\Lambda^*[\tilde z,\tilde q]).$$

The weak convergence $\iota^*[z_\gamma,q_\gamma] \rightharpoonup [\tilde z,\tilde q]$ and the ellipticity of the bilinear form associated to $A^{-1}$ yield the assertion.

**Step 7**: The normal cone property (OC3) is satisfied.

Owing to (OC1) and the results of the preceding steps we obtain for all $v=[z,q]\in H$ with $z\le 0, |q|_2\le\beta$,

$$\begin{aligned}
\langle\tilde\lambda,\iota^*v-\tilde v\rangle &= -\langle \Lambda A^{-1}\Lambda^*\tilde v,\iota^*v-\tilde v\rangle + \langle \hat w,\iota^*v-\tilde v\rangle\\
&= \lim_{\gamma\to+\infty}\left(-\langle\iota\Lambda A^{-1}\Lambda^*\iota^* v_\gamma, v-v_\gamma\rangle + \langle\iota\hat w, v-v_\gamma\rangle\right)\\
&= \lim_{\gamma\to+\infty}\left(\tfrac{1}{\gamma}\langle Bv_\gamma, v-v_\gamma\rangle + ([\mu_\gamma,\nu_\gamma], v-v_\gamma)_{L^2}\right)\\
&\le \limsup_{\gamma\to+\infty}\tfrac{1}{\gamma}\|v_\gamma\|_H\|B\|\|v\| + \limsup_{\gamma\to+\infty}([\mu_\gamma,\nu_\gamma],v-v_\gamma)_{L^2}\\
&= \limsup_{\gamma\to+\infty}([\mu_\gamma,\nu_\gamma],v-v_\gamma)_{L^2},
\end{aligned}$$

where the last equality follows from (A.4). We further verify that

$$\begin{aligned}
(\nu_\gamma, q-q_\gamma)_{L^2(\Omega)^d} &= \gamma\int_\Omega [|q_\gamma|_2-\beta_\gamma+\tfrac{\hat\nu}{\gamma}]^+\mathfrak{q}(q_\gamma)(q-q_\gamma)\,\mathrm{dx}\\
&\le \gamma\int_\Omega [|q_\gamma|_2-(\beta_\gamma-\tfrac{\hat\nu}{\gamma})]^+(\beta-|q_\gamma|_2)\,\mathrm{dx}\\
&\le \gamma\int_\Omega [|q_\gamma|_2-(\beta_\gamma-\tfrac{\hat\nu}{\gamma})]^+(\beta-(\beta_\gamma-\tfrac{\hat\nu}{\gamma}))\,\mathrm{dx}\\
&\le ||[|q_\gamma|_2-(\beta_\gamma-\tfrac{\hat\nu}{\gamma})]^+||_{L^2(\Omega)}(\gamma||\beta-\beta_\gamma||_{L^2(\Omega)}+||\hat\nu||_{L^2(\Omega)})\\
&\le K||[|q_\gamma|_2-(\beta_\gamma-\tfrac{\hat\nu}{\gamma})]^+||_{L^2(\Omega)}.
\end{aligned}$$

Now, the boundedness of $M^2_\gamma(q_\gamma)$ implies that

$$\lim_{\gamma\to+\infty} ||[|q_\gamma|_2-(\beta_\gamma-\tfrac{\hat\nu}{\gamma})]^+||_{L^2(\Omega)} \to 0.$$

Likewise, it holds that

$$\begin{aligned}
(\mu_\gamma, z-z_\gamma)_{L^2(\Gamma_c)} \le -(\mu_\gamma,z_\gamma)_{L^2(\Gamma_c)} &= -\gamma([z_\gamma+\tfrac{\hat\mu}{\gamma}]^+, z_\gamma+\tfrac{\hat\mu}{\gamma})_{L^2(\Gamma_c)} + \tfrac{1}{\gamma}(\mu_\gamma,\hat\mu)_{L^2(\Gamma_c)}\\
&\le \tfrac{1}{\gamma}||\mu_\gamma||_{L^2(\Gamma_c)}||\hat\mu||_{L^2(\Gamma_c)} \to 0 \text{ for } \gamma\to+\infty,
\end{aligned}$$

by the boundedness of $M^1_\gamma(z_\gamma)$. Consequently we obtain $\langle\tilde\lambda,\iota^*v-\tilde v\rangle\le 0$ for all $v=[z,q]\in H$ with $z\le 0, |q|_2\le\beta$. In virtue of Assumption 4.1 a density argument completes the proof. □

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